Fourier Duality as a Quantization Principle*

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The Weyl-Wigner prescription for quantization on Euclidean phase spaces makes essential use of Fourier duality. The extension of this property to more general phase spaces requires the use of Kac algebras, which provide the necessary background for the implementation of Fourier duality on general locally compact groups. Kac algebras – and the duality they incorporate – are consequently examined as candidates for a general quantization framework extending the usual formalism. Using as a test case the simplest non-trivial phase space, the half-plane, it is shown how the structures present in the complete-plane case must be modified. Traces, for example, must be replaced by their noncommutative generalizations – weights – and the correspondence embodied in the Weyl-Wigner formalism is no more complete. Provided the underlying algebraic structure is suitably adapted to each case, Fourier duality is shown to be indeed a very powerful guide to the quantization of general physical systems.

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1 Introduction

The complete quantum description of a physical system presupposes the identification of the space of observables, the scene of its dynamical evolution. In the classical description, this space corresponds to a subalgebra of the algebra $C^\infty(M)$ of infinitely differentiable functions on the phase space. When the latter is the linear space $\mathbb{R}^2$ (or $\mathbb{R}^{2n}$), the relationship between the quantum and the classical cases is well known, given as it is by the Weyl correspondence prescription. The correspondence assumes the existence (and knowledge) of a possible classical version of the system and of a general prescription – "quantization" – to transform the classical into the quantum description. The ultimate goal is to uncover some "grand principle", a rule providing directly the quantal description: given a system, we should be able to identify the observables and their space without the mediation of classical quantities. It would then be possible to describe even purely quantum systems, for which there are no classical limits. Such an objective is still far ahead and for the present time we are condemned to proceed from classical systems, trying to work up a general procedure of quantization from particular examples. This trial-and-error approach blends rigorous assumptions with inferences from previous case-study experience.

The Weyl prescription, in its original form [28], makes use of a very particular kind of duality, the Pontryagin duality which holds only when the underlying group of linear symplectomorphisms is Abelian. The Pontryagin dual of an Abelian group G is the space of characters, which is also an Abelian group, though not necessarily the same. Thus, the Euclidean spaces $\mathbb{R}^n$ and the cyclic groups $\mathbb{Z}_n$ are self-dual in this sense, but the circle and the group of integers are dual to each other. The link between these groups is provided by generalized two-way Fourier transforms mapping the L^1-space of one into the L^∞-space of the other, and for this reason we shall use the expression *Fourier duality* as a synonym to (eventually generalized) Pontryagin duality. In the Euclidean linear case, the Abelian group involved is formed by the translations on phase space, which is isomorphic to its own Fourier dual: the Fourier transforms of functions on $\mathbb{R}^2$ are functions on $\mathbb{R}^2$. Given the Fourier transform $\tilde{f}$ of a classical dynamical function f, the Weyl prescription yields the corresponding operator (q-number function) as a Fourier transform of $\tilde{f}$ with a projective operator kernel. A formal inverse procedure (first considered by Wigner [29]), involving an integration on an operator space, gives then the correspond-

ing c-number function [12]. These c-number functions and their Fourier transforms belong to twisted (noncommutative) algebras, different from the usual Abelian algebras of convolution and pointwise product. The twisted convolution on $L^1(\mathbb{R}^2)$ and the Moyal product on $L^\infty(\mathbb{R}^2)$ arise naturally from the projective operator product through the Weyl correspondence. Because of this "quantum" origin, the deformation of the usual algebras of functions on phase space they represent is considered a quantization [5].

The main difficulty comes from the fact that, for general systems, including those whose phase space is the Euclidean space, the group acting on phase space (a group of linear symplectomorphisms, here called *special canonical group*) is not Abelian and/or compact. On compact groups the integration implied in the Fourier transform is defined in a simple way, as there exists a unique Haar measure, which is both left- and right-invariant. Amongst non-compact groups, the existence of Haar measures is assured only for those which are locally compact, though in general the left-invariant and the right-invariant measures differ (when they happen to be equal, the group is said to be *unimodular*). Group-to-group duality is, however, restricted to the commutative case: the space dual to a non-Abelian group is no more a group, but an algebra. Duality must be understood no more as a relationship between groups, but as a relationship in a wider category. A fair formulation for the general locally compact case has been obtained in the early seventies, and led to the introduction of Kac algebras. These are Hopf-von Neumann algebras with peculiar generalized measures, called Haar weights. Actually, for non-unimodular locally compact groups, Fourier duality is only possible in the Kac algebra framework. We must abstract from groups to Kac algebras in order to have a Fourier duality. In this sense the usual Weyl correspondence is part of a highly nontrivial *projective duality* for the Abelian group $\mathbb{R}^2$, where an algebra generated no more by linear, but by projective operators comes into play [2].

Our objective here is to give a step further in the question of quantization through the study of these analytic-algebraic aspects. The algebraic facet is better known: it is necessary to resort to Hopf-von Neumann algebras. These algebras are, however, rather involved operator algebras, on which many different topologies and measures can be defined. The analytic facet lies precisely in the choice of the correct topology and measure. Our guiding idea will be the assumption of Fourier duality, which stands at the heart both of the Weyl quantization approach and of the group duality alluded to above. Since Fourier duality in its more general form is implemented in the Kac algebra structural frame [9], we argue that they are also able

to provide a generalized Weyl prescription for quantizing a phase space on which a separable locally compact type I group acts by symplectomorphisms.

We take as a test case the simplest non-trivial example of phase space: the half-plane. The fact that the configuration space $\mathbb{R}_+$ is Abelian, that the manifold of the special canonical group coincides with the phase space manifold and that there is no need to consider central extensions of this canonical group accounts for the relative simplicity of the example. Its non-triviality comes from the non-trivial properties of the group which, besides being non-Abelian and non-compact, is non-unimodular. These are important features, which bring to light the main difficulties of quantization on a general phase space. Specifically, this example also shows why the usual Weyl-Wigner quantization procedure does not generalize straightforwardly and does not always lead to a generalized Moyal bracket [21], or to a deformation of the algebra of functions on phase space. Although this example does not cover quantization on a general phase space, where the respective canonical group may have little to do with the space it acts [13], we believe the duality principle it illustrates can be generalized to quantization on any phase space to which an operator algebra can be associated, as it is done in [17].

We begin with an exposition of the classical picture on the half-plane in section 2. We use, in order to select a group on phase space, Isham's canonical approach which, though not quite general, is enough for the case in view. In the next section we give some details and classify the induced irreducible representations of the half-plane special canonical group, which is in fact a special parametrization of the affine group on the line (conversely, we show in the Appendix that the half-plane is the unique non-trivial homogeneous symplectic manifold of the affine group on the line). Since it seems that neither Hopf-von Neumann nor Kac algebras are structures quite familiar to the Physics community, we review them in a separate section. We emphasize those Kac algebras which are related to groups, in order to show how group duality is attained. At the end of that section we also show how to decompose the operator Kac algebra of a type I group according to its unitary dual. This is not found in the Kac algebra literature and will be essential to the interpretation of the Weyl formula in the duality framework. The half-plane case, used all along more as a gate into non-trivial aspects, is finally retaken for its own sake and given its finish in section 5. The whole treatment leads to a reappraisal of the reach and limitations of the Weyl-Wigner formalism as a guide for quantization on general phase spaces.

2 The Special Canonical Group

The phase space we want to quantize on is the half-plane $\mathbb{R}_+ \times \mathbb{R}$ the cotangent bundle of the configuration space given by the half-line $\mathbb{R}_+$. On this symplectic manifold we use the coordinates x and p, in terms of which the symplectic form, given as the derivative of the Liouville canonical 1-form, is

$$\omega = d\theta_o = dp \wedge dx, \qquad x \in \mathbb{R}_+,\, p \in \mathbb{R}.$$

The symplectic form implements, through the equation

$$i_{X_f}\omega = -df, \tag{1}$$

a homomorphism between the space of C^∞-functions (Hamiltonians) and the space of symplectic Hamiltonian vector fields, whose kernel are the constants. Since ω is non-degenerate, (1) can be solved for the vector fields and yields, in the above coordinates, $X_f = \partial_p f\, \partial_x - \partial_x f\, \partial_p$. The symplectic form also provides a Lie algebra structure on the C^∞-functions, as it defines the Poisson bracket by

$$\{f, g\} = -\omega(X_f, X_g), \tag{2}$$

which is isomorphic to the Hamiltonian vector fields Lie algebra through

$$[X_f, X_g] = -X_{\{f,g\}}.$$

We shall follow Isham [13] in the first steps. To quantize a phase space we start by looking for a finite dimensional (for simplicity) group whose elements act as symplectomorphisms, that is, preserve the symplectic structure. The action must be transitive, so as to avoid any lack of globality in the quantum description, and also (quasi-)effective. It is thus necessary to find a finite dimensional group G_{hp} under whose action the half-plane is a symplectic homogeneous G-space. The task can be simplified by proceeding as follows. Consider a group whose manifold is the configuration space, $(\mathbb{R}_+, \cdot)$, and make it act on a linear space so as to get at least an almost-faithful representation (a representation whose kernel is discrete). Take the action of $\mathbb{R}_+$ on $\mathbb{R}$ given by

$$\lambda \in \mathbb{R}_+ \mapsto R_\lambda(a) = a\lambda,\;\; a \in \mathbb{R}. \tag{3}$$

Construct the semi-direct product group $\mathbb{R}_+ \oslash \mathbb{R}$, with the product operation given by

$$(\lambda, a)(\rho, b) = (\lambda\rho, a + \phi_\lambda(b)),$$

where $\phi_\lambda(b) = R^*_{\lambda^{-1}}(b) = b/\lambda$ is the homomorphism on $\mathbb{R}$ given by the representation R^* contragradient to (3). The identity in $\mathbb{R}_+ \oslash \mathbb{R}$ is $(1, 0)$ and the inverse element of (λ, a) is given by $(\lambda^{-1}, \phi_\lambda^{-1}(-a))$, with $\phi_{\lambda^{-1}} = \phi_\lambda^{-1}$.

Considering the left action

$$l_{(\lambda,a)}(x, p) = (\lambda x, p/\lambda - a),$$

of this group on the space $\mathbb{R}_+ \times \mathbb{R}$, we see that $G_{hp} = \mathbb{R}_+ \oslash \mathbb{R}$ is formed by some special linear canonical transformations on the half-plane. Actually, we show in the Appendix, using Kirillov's orbits method [16], that the half-plane is the only non-trivial symplectic manifold canonically invariant by G_{hp}.

The Lie algebra $\mathcal{G}_{hp}$ of G_{hp} can be obtained from the group product with the help of the formula $e^{tA}e^{sB}e^{-tA}e^{-sB} = e^{ts[A,B] + \textit{ higher orders in } t,s}$ and is given on $\mathbb{R} \oplus \mathbb{R}$ by

$$[(l, a), (r, b)] = (0, ar - lb). \tag{4}$$

It is straightforward to realize this Lie algebra in terms of symplectic Hamiltonian vector fields on the half-plane. By the exponential mapping $(l, a) \mapsto (e^l, a) \in \mathbb{R}_+ \oslash \mathbb{R}$ we introduce the 1-parameter subgroups $t \mapsto (e^{lt}, 0)$, $s \mapsto (1, as)$, whose action on $\mathbb{R}_+ \times \mathbb{R}$,

$$l_{l,a}(x, p) = (e^{lt}x, e^{-lt}p - as)$$

is easily found to be generated by the symplectic Hamiltonian (right-invariant) vector fields

$$X_{l,a}(x, p) = lx\partial_x - (lp + a)\partial_p,$$

corresponding to the Hamiltonians $h_{l,a}(x, p) = ax + lxp$. On $C^\infty(\mathbb{R}_+ \times \mathbb{R})$ these Hamiltonians define a Poisson subalgebra by

$$\{h_{l,a}, h_{r,b}\} = h_{0,ar-lb}, \tag{5}$$

whose structure is identical to that of (4). We can then say that there is a faithful momentum mapping $J : T^*\mathbb{R}_+ \to \mathcal{G}^*_{hp}$, allowing the association of the pair $(l, a) \in \mathcal{G}_{hp}$ to the Hamiltonian

function $h_{l,a}$ by the duality pairing $\langle J(x,p),(l,a)\rangle = h_{l,a}(x,p)$. By this Lie algebra isomorphism we privilege the functions $h_{l,a}$ as a preferred class of observables to be quantized. Also because of this isomorphism, there is no need to central-extend $\mathcal{G}_{hp}$ as it happens in the complete-plane case. In other words, the cohomology space $H^2(\mathcal{G}_{hp},\mathbb{R}) \sim H^2(G_{hp},\mathbb{R})$ is trivial [13]. It is then possible to take the unitary irreducible linear representations of the special canonical group realized in terms of operators on a given Hilbert space and try to find an unbounded operator (representation generator) in correspondence with each preferred observable on the half-plane. In the next section we will provide the representations necessary to characterize such operators but, differently from Isham's approach, functions will be associated to bounded operators *a la* Weyl, which means that we shall work at the group representation level.

3 Induced Representations and the Unitary Dual

Irreducible unitary representations of semi-direct product groups are easily constructed via Mackey's induced representation theory (see [20, 4, 11, 23]). In this section we construct irreducible unitary representations of $G_{hp} = \mathbb{R}_+ \oslash \mathbb{R}$ by that method. We first note that a G_{hp} element $g = (\lambda, a)$ can be decomposed in its $\mathbb{R}_+$ and $\mathbb{R}$ parts according to

$$(\lambda, a) = (\lambda, 0)(1, \phi_\lambda^{-1}(a)) = g_{\mathbb{R}_+} g_{\mathbb{R}} \tag{6}$$

and

$$(\lambda, a) = (1, a)(\lambda, 0). \tag{7}$$

We begin by looking for unitary irreducible representations of the subgroup $\mathbb{R}$. This is immediate, since $\mathbb{R}$ is an Abelian normal subgroup. Its character (one-dimensional) representations on $\mathbb{C}$ are given by $V_x(a) = e^{ixa}$, where x is a label contained in the unitary dual group $\hat{\mathbb{R}} \sim \mathbb{R}$. The Hilbert space where our induced representation of G_{hp} will be realized is constructed as the space of functions $f : G_{hp} \to \mathbb{C}$ which can be decomposed into wavefunctions ξ, $f_x(g) = V_x^{-1}(g_{\mathbb{R}})\xi(g_{\mathbb{R}_+})$, or, using (6) with $g = (\lambda, a)$,

$$f_x(\lambda, a) = e^{-ix\phi_\lambda^{-1}(a)}\xi(\lambda), \qquad \xi \in L^2(\mathbb{R}_+), \tag{8}$$

and on which f_x satisfies

$$\int_{\mathbb{R}_+} \frac{d\lambda}{\lambda}\ |f_x(\lambda, a)|^2 = \int_{\mathbb{R}_+} \frac{d\lambda}{\lambda}\ |\xi(\lambda)|^2 < \infty.$$

We indicate this space by $H_x(G_{hp})$ and, in agreement with (8), use the fact that it is isomorphic to $L^2(\mathbb{R}_+)$. The induced representation of G_{hp} on $H_x(G_{hp})$ is then defined, for each $x \in \mathbb{R}$, by $[T_x(g)f_x](h) = f_x(g^{-1}h)$, or, directly in terms of wavefunctions in the coordinate representation (that is, on $L^2(\mathbb{R}_+)$), by

$$[T_x(g)\xi](h_{\mathbb{R}_+}) = V_x^{-1}([g^{-1}h_{\mathbb{R}_+}]_{\mathbb{R}})\xi([g^{-1}h_{\mathbb{R}_+}]_{\mathbb{R}_+}), \tag{9}$$

or still, even more explicitly, using $g = (\lambda, a)$, $h = (\rho, b)$, and computing $g^{-1}h_{\mathbb{R}_+} = (\lambda^{-1}\rho, \phi_\rho^{-1}(-a))$,

$$[T_x(\lambda, a)\xi](\rho) = e^{ix\phi_\rho^{-1}(a)}\xi(\lambda^{-1}\rho). \tag{10}$$

That these operators do represent the group G_{hp},

$$T_x(\lambda, a)T_x(\rho, b) = T_x((\lambda, a)(\rho, b)), \tag{11}$$

follows trivially from comparing the two identities below: applying the left hand side of (11) to $\xi \in L^2(\mathbb{R}_+)$, we obtain

$$\begin{aligned} [T_x(\lambda, a)T_x(\rho, b)\xi](\eta) &= e^{ix\phi_\eta^{-1}(a)}[T_x(\rho, b)\xi](\lambda^{-1}\eta) \\ &= e^{ix\phi_\eta^{-1}[a+\phi_\lambda(b)]}\xi((\lambda\rho)^{-1}\eta), \end{aligned}$$

while the right hand side gives

$$\begin{aligned} [T_x((\lambda, a)(\rho, b))\xi](\eta) &= [T_x(\lambda\rho, a + \phi_\lambda(b))\xi](\eta) \\ &= e^{ix\phi_\eta^{-1}[a+\phi_\lambda(b)]}\xi((\lambda\rho)^{-1}\eta). \end{aligned}$$

Unitarity and irreducibility of the representation (10) will be proved in the following.

Abstracting from the Hilbert space $L^2(\mathbb{R}_+)$, we can write the operator T_x as

$$T_x(\lambda, a)|_\rho = e^{ix\phi_{\hat{\rho}}^{-1}(a)}e^{-i\ln(\lambda)\hat{\pi}},$$

where $|_\rho$ means that the operator acts on the argument ρ in such a way that the multiplication and dilation operators are defined by

$$\begin{aligned}\hat{\rho}\xi(\rho) &= \rho\xi(\rho)\\ \hat{\pi}\xi(\rho) &= -i\rho\partial_\rho\xi(\rho).\end{aligned}$$

An operatorial version of the decompositions given at the beginning of this section comes up if we define the operators (dropping $|_\rho$ from now on)

$$T_x(\lambda,0) \equiv L(\lambda) = e^{-i\ln(\lambda)\hat{\pi}}; \tag{12a}$$

$$T_x(1,a) \equiv \hat{V}_x(a) = e^{ix\phi_{\hat{\rho}}^{-1}(a)}, \tag{12b}$$

with which we have

$$T_x(\lambda,a) = \hat{V}_x(a)L(\lambda), \tag{13}$$

where $L(\lambda)$ is identified as the left-regular unitary representation of the group $(\mathbb{R}_+,\cdot)$ on $L^2(\mathbb{R}_+)$. Definitions (12) also allows us to rewrite (6) and (7) in operatorial form

$$L(\lambda)\hat{V}_x(\phi_\lambda^{-1}(a)) = \hat{V}_x(a)L(\lambda). \tag{14}$$

Expanding the identity above according to (12), and recalling that $\phi_\lambda(a) = a/\lambda$, we obtain, up to first order in $al,\ l = \ln\lambda$,

$$[\hat{\rho},\hat{\pi}] = i\hat{\rho}.$$

Now the unitarity of (10) follows easily from (13) and the unitarity of L and V_x,

$$\begin{aligned}T_x^\dagger(\lambda,a) &= L(\lambda^{-1})\hat{V}_x(-a)\\ &= \hat{V}_x(\phi_\lambda^{-1}(-a))L(\lambda^{-1})\\ &= T_x((\lambda,a)^{-1}),\end{aligned}$$

where the second equality comes from (14).

At this point we should ask whether there exists an equivalence relation between the operators T_x. This is an important question if we want to do harmonic analysis on the group $G_{hp} = \mathbb{R}_+ \oslash \mathbb{R}$, as we shall, for we must sum (integrate) over the unitary dual $\widehat{G_{hp}}$ of G_{hp},

the space of classes of inequivalent irreducible representations. To answer the question we begin by observing that the right-regular representation of $(\mathbb{R}_+, \cdot)$ acts on $\xi \in L^2(\mathbb{R}_+)$ by $[R(\rho)\xi](\eta) = \xi(\eta\rho)$. In order to verify whether this operator is an intertwining for the T_x, we calculate

$$\begin{aligned} [R(\rho)^{-1}T_x(\lambda, a)R(\rho)\xi](\eta) &= [T_x(\lambda, a)R(\rho)\xi](\eta\rho^{-1}) \\ &= e^{ix\phi^{-1}_{\eta\rho^{-1}}(a)}\xi(\lambda^{-1}\eta). \end{aligned} \tag{15}$$

Now, remembering that $\mathbb{R}_+$ acts on $\mathbb{R}$ by ϕ_ρ, its associated co-action $\hat{\phi}_\rho$ on the character space $\hat{\mathbb{R}}$ is defined by

$$[\hat{\phi}_\rho\chi_x](a) \equiv \chi_x(\phi_\rho^{-1}(a)) = e^{ix\phi_\rho^{-1}(a)}.$$

With this at hand, it is easy to see that the coefficient of the wavefunction ξ in (15) above is just the following co-action:

$$\begin{aligned} [\hat{\phi}_{\eta\rho^{-1}}\chi_x](a) = \chi_x(\phi^{-1}_{\eta\rho^{-1}}(a)) &= \chi_x(\phi_\rho \circ \phi_\eta^{-1}(a)) \\ &= \hat{\phi}_{\rho^{-1}}\chi_x(\phi_\eta^{-1}(a)). \end{aligned}$$

Explicitly, considering that $\chi_x \in \hat{\mathbb{R}} \sim \mathbb{R} \ni x$, we have the co-action given by

$$\hat{\phi}_{\rho^{-1}}(x) = \rho^{-1}x. \tag{16}$$

We then conclude that the right-regular representation R is an intertwining operator for the T's, connecting them by the co-action $\hat{\phi}$,

$$R(\rho)^{-1}T_x(\lambda, a)R(\rho) = T_{\hat{\phi}_{\rho^{-1}}(x)}(\lambda, a). \tag{17}$$

We have then three classes of representations: one for $x > 0$ and one for $x < 0$, both isomorphic to $\mathbb{R}_+$; and that one represented by the point $x = 0$. We shall indicate the cases $x > 0$ or $x < 0$ simply by $\pm$ and write the two infinite-dimensional representation operators as

$$T_\pm(\lambda, a) = e^{\pm ia\hat{p}}L(\lambda). \tag{18}$$

In the case $x = 0$, we have simply $L(\lambda)$, the left-regular unitary representation of $(\mathbb{R}_+, \cdot)$, $T_0(\lambda, a) = L(\lambda)$. This representation is reducible, that is, it is possible to decompose it in terms of the $(\mathbb{R}_+, \cdot)$ characters $\chi_y(\lambda) = \lambda^{iy},\ y \in \mathbb{R}$, and write formally

$$L = \int_{\mathbb{R}}^{\oplus} dy\, \chi_y.$$

This give us an infinity ($\mathbb{R}$) of 1-dimensional irreducible representations,

$$T_y(\lambda, a) = \lambda^{iy}. \tag{19}$$

Summing up: once we suppose the irreducibility of the T_x, which will be proved just below, the unitary dual $\widehat{G_{hp}}$ is given by $\{+\} \cup \{-\} \cup \mathbb{R}$. If we compare this result with the orbits of the coadjoint action of G_{hp} obtained in the Appendix, we observe that the formula $\widehat{G_{hp}} = \mathcal{G}^*_{hp}/G_{hp}$ holds.

Now, to address the problem of irreducibility of the induced representations T_x we shall refer to an important result of Mackey's theory. Mackey's imprimitivity theorem [4, 25] for semi-direct products states that the induced representations of such groups, in our case T_x, will be irreducible if and only if the semi-direct product group $\mathbb{R}_+ \oslash \mathbb{R}$ which it represents satisfies a condition of *regularity*. This condition essentially means that the $\mathbb{R}_+$-orbits in $\hat{\mathbb{R}}$ by the $\hat{\phi}$ action are countably separated with respect to the Borel structure. This is easily seen to be fulfilled since $\hat{\mathbb{R}} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+$. So, our group G_{hp} is regular and the representations T_x are irreducible.

The above analysis gives still another important information about the group G_{hp}. Type I groups are, roughly speaking, those groups which have a well behaved Borel structure on the unitary dual, more specifically, the decomposition of representations of these groups into irreducible representations is unique [18]. Good examples are the Abelian and the semi-simple groups. From another theorem by Mackey [4, p. 536], a regular semi-direct product group, say $\mathbb{R}_+ \oslash \mathbb{R}$, is a *type I group* if and only if for each $x \in \hat{\mathbb{R}}$, its isotropy subgroup I_x is a type I group. Well, we know that the orbits through x are given by $\mathcal{O}_x = \mathbb{R}_+/I_x$, and we have found that they are isomorphic either to $\mathbb{R}_+$ $(\mathbb{R}_+, \cdot)$ or to the trivial $\{e\}$. Consequently, each isotropy subgroup is necessarily isomorphic to one of them, and they are both of type I.

4 Kac Algebras and Group Duality

Once characterized and constructed the representations of the group under which the half-plane is canonically invariant, we must give a rule to associate an operator to each observable. To do it we will use the powerful techniques provided by the Kac algebras. These algebras were constructed in 1973 independently by G.I. Kac and L.I. Vaĭnermann [14], and M. Enock and J.-

M. Schwartz [8], with the objective of generalizing to non-unimodular locally compact groups the Pontryagin (Abelian groups) and Tannaka-Krein (compact groups) duality theorems. A duality for locally compact (l.c. from now on) non-unimodular groups, comprising previous works of P. Eymard, N. Tatsuuma and J. Ernst on a category wider than that of such groups, had already been partially obtained in the seventies by M. Takesaki [24] in the Hopf-von Neumann algebra framework. Unfortunately, due to an incomplete theory of noncommutative integration, Takesaki's work on that direction had a lack of symmetry. A general duality only was possible after a considerable knowledge on weights was obtained. This knowledge led to the definition of Kac algebras by the addition of a suitable (Haar) weight on Hopf-von Neumann algebras.

Actually, a general duality for locally compact groups is achieved if we associate to them two Kac algebras, one on the von Neumann algebra of L^∞-functions and the other on the von Neumann algebra generated by left-regular representations. These two algebras turn out to be dual in the category of Kac algebras. This means that, by duality, to each L^∞-function on the group we can make to correspond an operator written as a linear combination of the left-regular representations. In this section we will introduce Hopf-von Neumann and Kac algebras, apply the latter to groups in order to show the l.c. group duality, and show how they decompose following the unitary dual of a type I group.

Since Hopf-von Neumann and Kac algebras are, to begin with, von Neumann algebras, we start by recalling some definitions on these algebras which will be necessary (see, for example Ref. [6]). A *von Neumann algebra* M is an *involutive unital* subalgebra of the Banach algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H, closed with respect to the *strong topology*, a topology which is defined by the open balls of the family of *seminorms*

$$\|a\|_{F,\psi} = \|a\psi\|, \ \ \psi \in H. \tag{20}$$

Besides this topology on $\mathcal{B}(H)$ there is also the *uniform* topology, which is defined through the norm

$$\|a\| = \sup\{\|a\psi\|_H, \ \|\psi\|_H = 1, \ \psi \in H\}. \tag{21}$$

As a map $\| \ \| : M \to [0, \infty]$, this norm satisfies the following conditions:

- $\|a\| = 0$ if and only if $a = 0$;

- $\|a+b\| \leq \|a\| + \|b\|$;
- $\|\alpha a\| = |\alpha| \|a\|, \qquad \alpha \in \mathbb{C}$;
- $\|ab\| \leq \|a\| \, \|b\|$.

The first axiom does not hold for a *seminorm*. We shall consider also an another topology coming from seminorms, the *ultra*(σ)-weak topology. It is given by $\|T\|_{\sigma,\psi_i,\phi_i} = \sum_i |(T\psi_i, \phi_i)|$, where $\psi_i, \phi_i \in H$ are such that $\sum_i \|\psi_i\|^2 < \infty$ and $\sum_i \|\phi_i\|^2 < \infty$. The *predual* M_* of M is the (Banach) space of the ultra-weakly continuous linear functionals on M.

The word *involutive*, used in the definition above, means that on M is defined a map $* : M \to M$, the *involution*, such that

- $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$;
- $(ab)^* = b^* a^*$;
- $(a^*)^* = a$.

Besides these axioms, on a von Neumann algebra it is also true that $\|a^*\| = \|a\|$ (it is an involutive Banach algebra), $\|a^* a\| = \|a\|^2$ (it is a C^*-algebra) and that the unit is preserved by the involution, $\mathbf{1}^* = \mathbf{1}$. Finally, a W^* algebra is an algebra M which equals the dual of its predual, $M = (M_*)^*$. It is, roughly speaking, an abstract C^*-algebra which can always be realized as a von Neumann algebra on a suitable Hilbert space H.

We can introduce at this point the definition of a Hopf-von Neumann algebra. A *co-involutive Hopf-von Neumann algebra* is a triple $\mathbb{H} = (M, \triangle, \kappa)$ where [9]

- M is a W^*-algebra;
- the homomorphism $\triangle : M \to M \otimes M$, called *coproduct*, is normal, injective and such that

$$\triangle \mathbf{1} = \mathbf{1} \otimes \mathbf{1} \tag{22a}$$

$$(\triangle \otimes id) \circ \triangle = (id \otimes \triangle) \circ \triangle. \tag{22b}$$

The first statement above says that $\triangle$ is unital and the latter that it is coassociative. Since $\triangle$ is a homomorphism of W^*-algebras, this means that it is linear and

$$\triangle(ab) = \triangle(a) \triangle(b). \tag{23}$$

- there is a map $\kappa : M \to M$, called *co-involution*, which is an involutive anti-automorphism, that is, which is linear and such that, $\forall a, b \in M$,

$$\kappa(ab) = \kappa(b)\kappa(a); \tag{24a}$$
$$\kappa(a^*) = \kappa(a)^*; \tag{24b}$$
$$\kappa(\kappa(a)) = a. \tag{24c}$$

- it is also an anti-coautomorphism,

$$(\kappa \otimes \kappa) \circ \triangle = \sigma \circ \triangle \circ \kappa, \tag{25}$$

  where $\sigma(a \otimes b) = b \otimes a$.

$\mathbb{H}$ is said to be Abelian or commutative if M is Abelian, and symmetric or cocommutative if $\sigma \circ \triangle = \triangle$. Note that from (24c) and (24b) it follows that $\kappa(\kappa(a^*)^*) = a$, or $\kappa \circ * \circ \kappa \circ * = id$, but the converse is not true. This condition is actually weaker than those axioms. One of the differences between Hopf-von Neumann algebras and Woronowicz's "compact matrix pseudo-groups" [30] is that this weaker condition is imposed instead of (24c), (24b).

Given a co-involutive Hopf-von Neumann algebra $(M, \triangle, \kappa)$, where M acts on the Hilbert space H, and a representation μ of its predual M_* on the Hilbert space H_μ, a partial isometry $U \in \mathcal{B}(H_\mu) \otimes M$ such that

$$\mu(\omega) = (id \otimes \omega)(U), \qquad \omega \in M_*, \tag{26}$$

is said to be the *generator* of μ. If μ is *multiplicative* and *involutive*, its generator U satisfies the respective identities

$$(id \otimes \triangle)(U) = (U \otimes \mathbf{1})(\mathbf{1} \otimes \sigma)(U \otimes \mathbf{1})(\mathbf{1} \otimes \sigma) \tag{27a}$$
$$(id \otimes \omega \circ \kappa)(U) = (id \otimes \omega)(U^*). \tag{27b}$$

In the following we shall also denote $(U \otimes \mathbf{1})$ and $(\mathbf{1} \otimes \sigma)(U \otimes \mathbf{1})(\mathbf{1} \otimes \sigma)$ in $\mathcal{B}(H_\mu) \otimes M \otimes M$ by U_{12} and U_{13} respectively. If $\xi, \eta \in H$ we define the linear form $\omega_{\xi,\eta} \in M_*$ by

$$\langle a, \omega_{\xi,\eta} \rangle \equiv (a\xi|\eta)_H, \qquad \forall a \in M. \tag{28}$$

The formula

$$(\hat{U}(\alpha \otimes \beta)|\gamma \otimes \delta)_{H \otimes H_\mu} = (\beta|\mu(\omega_{\gamma,\alpha})\delta)_{H_\mu}, \qquad \alpha, \gamma \in H,\ \beta, \delta \in H_\mu, \tag{29}$$

coming from (26) and (28), and connecting the representation μ and the operator $\hat{U} \equiv \sigma \circ U^* \circ \sigma \in M \otimes \mathcal{B}(H_\mu)$ (the dual of U), will be very useful.

Before introducing Kac algebras, some facts concerning weights and the representation of a von Neumann algebra by a weight – the GNS construction – are worth mentioning. The basic references are [6] and [9, section 2.1].

Consider a map from the set of strictly positive elements of M, $\varphi : M^+ \to [0, \infty]$, with the conditions

- $\varphi(a + b) = \varphi(a) + \varphi(b)$;
- $\varphi(\lambda a) = \lambda \varphi(a),\ \forall \lambda \geq 0,$ where $0 \cdot \infty \equiv 0$;
- $\varphi(a^* a) = \varphi(a a^*)\ \forall a \in M$.

The first two conditions define a *weight* on M, and the three together define a *trace* [7]. A weight generalizes the concept of positive linear functional on C*-algebras and, in particular, the concept of *state*. Associated to φ we define the left ideal $\mathcal{N}_\varphi \subset M$ by $\{a \in M \,|\, \varphi(a^* a) < \infty\}$, and the involutive algebra $\mathcal{M}_\varphi$ as the linear span of $\{a \in M^+ \,|\, \varphi(a) < \infty\} \subseteq \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, with $\mathcal{N}_\varphi^* = \{a^* \mid a \in \mathcal{N}_\varphi\}$. A weight φ is called:

normal if for every sequence $\{a_i\}$ with upper bound $a \in M^+$, $\varphi(a)$ is the upper bound of the sequence $\{\varphi(a_i)\}$;

faithful if $\varphi(a) = 0 \Rightarrow a = 0,\ a \in M^+$,

semi-finite if $\mathcal{M}_\varphi$ is ultra-weakly dense in M.

Given a normal, faithful and semi-finite weight φ on a von Neumann algebra M, we construct a representation of M by the following procedure [6]: φ defines a scalar product in $\mathcal{N}_\varphi$, through

$$(a|b)_\varphi \equiv \varphi(b^* a).$$

It is actually only a quasi-scalar product since, as $\varphi(a^* a) \geq 0,\ (a|a)_\varphi$ can be zero. To circumvent this problem, we should factor the left ideal $I_\varphi = \{b \in A \mid (b|b)_\varphi = 0\}$ out of $\mathcal{N}_\varphi$. The quotient

is formed by equivalence classes [b] of elements b' such that b-b' is in I_φ. $\mathcal{N}_\varphi/I_\varphi$ has a pre-Hilbert structure given by the scalar product $([a],[b]) = (a|b)_\varphi$ which is invariant on each class. Completing $\mathcal{N}_\varphi/I_\varphi$ with respect to this product we get the Hilbert space H_φ. The map

$$\begin{aligned} \pi_\varphi(a) : \mathcal{N}_\varphi/I_\varphi &\rightarrow \mathcal{N}_\varphi/I_\varphi \\ [b] &\mapsto [ab] \end{aligned}$$

is bounded and can be extended to H_φ as a bounded operator. We call (π_φ, H_φ) the *GNS construction* of (M,φ), and a_φ denotes the image of $a \in \mathcal{N}_\varphi$ into H_φ by the canonical injection $\pi_\varphi : \mathcal{N}_\varphi \rightarrow H_\varphi$, $a \mapsto \pi_\varphi(a) = [a]$. The image $\pi_\varphi(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*)$ is a *left Hilbert algebra* [9], which is isomorphic to M. The image of the involution $*$ is the operator S_φ, which has the polar decomposition

$$S_\varphi = J_\varphi \triangle_\varphi^{1/2}.$$

This decomposition gives rise to the antilinear isometry $J_\varphi : H_\varphi \rightarrow H_\varphi$, such that $JMJ = M'$, $JaJ = a^*$, $a \in Z(M)$, and to the *modular operator* $\triangle_\varphi$, where $M' = \{a \in \mathcal{B}(H) \mid ab = ba, for\ all\ b \in M\}$ is the *commutant* of M, and $Z(M) = M \cap M'$ is the *center* of M. The modular operator satisfies $\triangle_\varphi^{it} M \triangle_\varphi^{-it} = M$, for all $t \in \mathbb{R}$, and this leads to the definition of the *modular automorphism group* σ_t^φ on M by

$$\sigma_t^\varphi(a) = \triangle_\varphi^{it}\, a\, \triangle_\varphi^{-it}. \tag{30}$$

The modular group σ_t^φ is such that the weight φ is invariant, $\varphi = \varphi \circ \sigma_t^\varphi$, and is also characterized by the fact that φ is the unique KMS-weight associated to it. This short overview on the Tomita-Takesaki theory extended to weights will be enough to introduce Kac algebras.

A *Kac algebra* $\mathbb{K} = (M, \triangle, \kappa, \varphi)$ satisfies the following axioms (for a good review on Kac algebras, see the first sections of an article by one of its founders in Ref. [26]):

- $(M, \triangle, \kappa)$ is a co-involutive Hopf-von Neumann algebra;
- $\varphi : M^+ \rightarrow [0, \infty]$ is a normal, faithful and semi-finite weight on M called *Haar weight* such that

– $\triangle(\mathcal{N}_\varphi) \subset \mathcal{N}_{\mathbf{1}\otimes\varphi}$. A stronger version of this axiom is more manageable and will be used. It says that φ is left-invariant with respect to $\triangle$, or

$$(id \otimes \varphi)\triangle(a) = \varphi(a)\mathbf{1} \qquad \forall a \in M^+; \tag{31}$$

– φ is symmetric, $\forall\ a, b \in \mathcal{N}_\varphi$,

$$(id \otimes \varphi)[(\mathbf{1} \otimes b^*)\triangle(a)] = \kappa \circ (id \otimes \varphi)[\triangle(b^*)(\mathbf{1} \otimes a)]; \tag{32}$$

– and

$$\kappa \circ \sigma_t^\varphi = \sigma_{-t}^\varphi \circ \kappa \qquad \forall\ t \in \mathbb{R}. \tag{33}$$

Given a Hopf-von Neumann algebra $\mathbb{H}$, then a Haar weight which makes of $\mathbb{H}$ a Kac algebra, if it exists, is unique up to a scalar [9, sect. 2.7.7]. A Kac algebra $\mathbb{K}$ is called *unimodular* if the Haar weight φ is a trace and is invariant by κ, $\varphi = \varphi \circ \kappa$. When φ is a trace, then it is true that $\triangle_\varphi = 1$ and $\sigma_t^\varphi = id.$, as it happens for the Abelian Kac algebras of groups described in the next subsection.

Associated to a Kac algebra there exists always an isometry W belonging to $M \otimes \mathcal{B}(H_\varphi)$, called the *fundamental operator*, such that

$$W(a_\varphi \otimes b_\varphi) = [\triangle(b)(a \otimes \mathbf{1})]_\varphi \qquad a, b \in \mathcal{N}_\varphi. \tag{34}$$

This unitary operator implements the coproduct as follows:

$$\triangle(a) = W(\mathbf{1} \otimes a)W^*. \tag{35}$$

Let us now introduce the dual of a Kac algebra $\mathbb{K}$ based on M. Its predual M_* has a product $*$ given by

$$\langle a, \omega * \omega' \rangle = \langle \triangle a, \omega \otimes \omega' \rangle, \tag{36}$$

and an involution ^{o} by

$$\langle a, \omega^o \rangle = \overline{\langle \kappa(a)^*, \omega \rangle}. \tag{37}$$

The predual is thus an involutive Banach algebra. Besides the GNS representation of M on H_φ, there is a multiplicative and involutive representation of M_* on the same Hilbert space,

$$\lambda : M_* \to \hat{M} \subset \mathcal{B}(H_\varphi).$$

The representation λ is such that $\lambda(\omega)$ is a bounded operator on $\pi_\varphi(\mathcal{N}_\varphi)$ which acts on H_φ by

$$\lambda(\omega)(a_\varphi) = [(\omega \circ \kappa \otimes id)\triangle(a)]_\varphi,$$

and can also be written

$$\lambda(\omega) = (\omega \circ \kappa \otimes id)(W). \tag{38}$$

λ is called the *Fourier representation* of $\mathbb{K}$. Its image $\lambda(M_*) = \hat{M}$ is a von Neumann algebra on which it is true that

$$\lambda(\omega)(\omega'_\varphi) = (\omega * \omega')_\varphi, \tag{39}$$

where $\omega'_\varphi \in H_\varphi$ is the unique vector such that $(\omega'_\varphi|a_\varphi) = \langle a^*, \omega' \rangle$, for all $a \in \mathcal{N}_\varphi$, and for every $\omega' \in I_\varphi = \{\omega \in M_* \,|\, sup\{|\langle \omega, a^* \rangle|,\ a \in M,\ \varphi(a^* a) \leq 1\} < \infty\}$. Actually, the above condition together with the definition of I_φ is a generalization of the condition of square integrability for $\omega' \in M_*$.

The dual of the Kac algebra $\mathbb{K}$ is based on the image $\hat{M}$ of the Fourier representation, $\hat{K} = (\hat{M}, \hat{\triangle}, \hat{\kappa}, \hat{\varphi})$. The dual involution ^{o} goes to $\hat{S}_{\hat{\varphi}} = \hat{J}_{\hat{\varphi}} \hat{\triangle}_{\hat{\varphi}}^{1/2}$, analogously to its dual. The operator W is unitary and its adjoint is given by (with $\hat{J}_{\hat{\varphi}} = \hat{J}$ from now on)

$$W^* = (\hat{J} \otimes J) \circ W \circ (\hat{J} \otimes J). \tag{40}$$

Its dual is then written

$$\hat{W} = \sigma \circ W^* \circ \sigma, \tag{41}$$

and the dual coproduct is given by the dual version of (34), or under the form

$$\hat{\triangle}(\omega) = \hat{W}(\mathbf{1} \otimes \omega)\hat{W}^*. \tag{42}$$

Furthermore, the dual antipode is defined by $\hat{\kappa}(\lambda(\omega)) = \lambda(\omega \circ \kappa)$ or by $\hat{\kappa}(\omega) = J\omega^o J$, its canonical implementation on H_φ. Dualizing this last formula, we get a new formula for κ in terms of $\hat{J}$:

$$\kappa(a) = \hat{J} a^* \hat{J}. \tag{43}$$

The dual weight $\hat{\varphi}$ on $\hat{M}$ is the normal, faithful semi-finite weight canonically associated to the left Hilbert algebra $(I_\varphi \cap I_\varphi^o)_\varphi$ and is given, for $X \in \hat{M}^+$, by [9, 2.1.1 and 3.5.3.]

$$\hat{\varphi}(X) = \begin{cases} \|\omega\|^2 & \text{if there exists } \omega \in (I_\varphi \cap I_\varphi^o)_\varphi \text{ such that } X = \hat{\pi}(\omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\hat{\pi}$ is the canonical representation (acting on the left by the algebra product) of $(I_\varphi \cap I_\varphi^o)_\varphi$ on H_φ. Finally, the Hilbert space $H_{\hat{\varphi}}$ is identified with H_φ. This Kac algebra obviously has also a predual $\hat{M}_*$ and a Fourier representation $\hat{\lambda}$.

From (38) and by the fact that λ is an involutive representation, (27b), it follows by using (41) that

$$\lambda(\omega) = (\omega \otimes id)(W^*) = (id \otimes \omega)(\hat{W}).$$

If we compare this formula with (26), we get $\hat{W}$ as the generator of λ. Furthermore, if we apply (27a) to $\hat{W}^* = \sigma \circ W \circ \sigma$, we obtain $(\triangle \otimes id)(W) = (\mathbf{1} \otimes W)(\sigma \otimes \mathbf{1})(\mathbf{1} \otimes W)(\sigma \otimes \mathbf{1})$, and from the relation (35) it follows that W satisfies the *pentagonal relation*

$$(\mathbf{1} \otimes W)(\sigma \otimes \mathbf{1})(\mathbf{1} \otimes W)(\sigma \otimes \mathbf{1})(W \otimes \mathbf{1}) = (W \otimes \mathbf{1})(\mathbf{1} \otimes W).$$

The duality mapping between $\mathbb{K}$ and $\hat{\mathbb{K}}$ is then performed first by passing from M to its predual M_* and then to $\hat{M}$ via the Fourier representation λ, which is faithful [9, Chap. 4]. Since λ is generated by $\hat{W}$ (and, by duality, $\hat{\lambda}$ is generated by W), we understand the essential role played by the operator W in Kac duality. The Kac duality is then the fact that $\hat{\hat{\mathbb{K}}}$ is isomorphic to $\mathbb{K}$.

4.1 The Abelian Kac Algebra of a Group

Given a separable l.c. group G, there are two Kac algebras of special significance. The first is defined on the von Neumann algebra $L^\infty(G)$ of (classes of almost everywhere defined) measurable and essentially bounded functions on G [15] This means that, for every $f \in L^\infty(G)$, there

exists a smallest number C $(0 \leq C < \infty)$ such that $|f(x)| \leq C$ locally almost everywhere. This number C is just the $ess.sup.$ (essential supremum) of f. The norm is given by

$$\|f\|_\infty = ess.sup.|f(x)|$$

and the involution by $f^*(x) = \overline{f(x)}$. This algebra acts on the Hilbert space $L^2(G)$ by pointwise multiplication. This Hilbert space has the L^2 scalar product

$$(f|g) = \int_G dx\, f(x)\overline{g(x)},$$

and the norm $\|f\|_2 = \sqrt{(f|f)}$, where dx is the left invariant Haar measure on G. Then $L^\infty(G)$, with the operations and conditions of the following list, is the *Abelian Kac algebra* of G, $\mathbb{K}^a(G) = (L^\infty(G), \triangle, \kappa, \varphi_a)$:

$$f \cdot g(x) = f(x)g(x); \tag{44a}$$

$$\mathbf{1} = 1, \text{ such that } 1(x) = 1\ \forall\ x \in G; \tag{44b}$$

$$\triangle(f)(x,y) = f(xy); \tag{44c}$$

$$\kappa(f)(x) = f(x^{-1}); \tag{44d}$$

$$\varphi_a(f) = \int_G dx\ f(x), \qquad f \in L^\infty(G)^+. \tag{44e}$$

Here $C(G)$ is the algebra of continuous functions with compact support on G, whose product is the convolution (see below). The Haar weight is in fact a trace, simply the left invariant Haar measure on G. In consequence, the modular operator is reduced to simple multiplication by 1, $\triangle_{\varphi_a} = 1$, and the modular group is trivial, $\sigma_t^{\varphi_a} = id$. The underlying Abelian Hopf-von Neumann algebra is denoted $\mathbb{H}^a(G)$.

We also have in this case, for $F \in C(G \otimes G)$ and $f \in C(G)$,

$$WF(x,y) = F(x,xy), \qquad W^*F(x,y) = F(x,x^{-1}y),$$
$$\hat{J}f(x) = \frac{1}{\sqrt{\triangle_G x}}\overline{f(x^{-1})},$$

where $\triangle_G$ is the modular function on G to be defined a few steps below. From these data we can use relation (29) to compute the Fourier representation for $\mathbb{K}^a(G)$,

$$\begin{aligned}(W(f\otimes g)|h\otimes l) &= \int_G dx\, f(x)\overline{h(x)} \int_G dy\, g(xy)\overline{l(y)} \\ &= \int_G dz\, g(z) \int_G dx\, \overline{h(x)}f(x)\overline{l(x^{-1}z)}, \\ &= (g|\lambda(\omega_{hf})l).\end{aligned}$$

We conclude that $\lambda(\omega_{hf})l(z) = \int_G dx\, h(x)\overline{f(x)}l(x^{-1}z)$ or, taking into account that $\omega_{h,f} = h\overline{f} \in L^\infty(G)_*$ by (28), that

$$\lambda(\omega_{hf}) = \int_G dx\, \omega_{hf}(x)L(x). \tag{45}$$

Since $\mathbb{K}^a(G)$ acts on $L^2(G)$, it follows from $\varphi_a(f^*f) = \int_G dx\ |f(x)|^2 < \infty$ that the GNS construction is given by inclusion, with $\mathcal{N}_{\varphi_a} = L^\infty(G) \cap L^2(G)$ and $\mathcal{M}_{\varphi_a} = L^\infty(G) \cap L^1(G)$. The predual is $L^\infty(G)_* = L^1(G)$ and, as anticipated, $I_{\varphi_a} = L^1(G) \cap L^2(G)$ is the space of square integrable functions on the predual $L^1(G)$, on which we now concentrate.

Given a left invariant Haar measure dx on a l.c. group G, the space of (classes of) functions defined almost everywhere and integrable on G, $L^1(G, dx)$, is the *convolution Banach algebra* of G with as product the convolution [22]

$$(f * g)(x) = \int_G dy\ f(y)g(y^{-1}x), \tag{46}$$

involution

$$f^*(x) = \triangle_G x^{-1}\overline{f(x^{-1})} \tag{47}$$

and norm $\|f\| = \int_G dx\ |f(x)|$. $\triangle_G : G \to \mathbb{R}_+$ is a positive and continuous homomorphism of groups called *modular function*:

$$\begin{aligned} \triangle_G e &= 1 \\ \triangle_G(xy) &= \triangle_G x \triangle_G y. \end{aligned}$$

If μ_l and μ_r are left- and right-invariant (Haar) measures on G, that is, $\mu_l(xE) = \mu_l(E)$, $\mu_r(Ey) = \mu_r(E)$, for every Borel set E, the function $\triangle_G$ relates them by the Radon-Nikodým derivative [22]

$$\frac{d\mu_l(x)}{d\mu_r(x)} = \triangle_G x. \tag{48}$$

When $\triangle_G \equiv 1$, the two measures coincide and G is *unimodular*. Changing variables in (46) and using the identity

$$\frac{d\mu_l(x^{-1})}{d\mu_l(x)} = \triangle_G x^{-1}, \tag{49}$$

the convolution can also be written in terms of the right invariant measure (see (48)) as $(f * g)(x) = \int_G d\mu_r(y)\, f(xy^{-1})g(y)$.

The algebra $C(G)$ of continuous functions with compact support is dense in $L^1(G)$, which explains its appearance in some definitions. G is discrete if and only if L^1 has a unit. Otherwise, it has only left and right approximate units. In general, the algebra $L^1(G)$ is nothing more than an ideal in the following unital algebra. To every $f \in L^1(G)$ we associate a measure $d\mu(x)$ by $d\mu(x) = f(x)dx$. This association implements an involutive isometry between the Banach algebras $L^1(G)$ and $M^1(G)$, the unital involutive algebra of all bounded complex Borel measures on G with convolution given by $(\mu * \nu)(f) = \int_{G\times G} f(xy)d\mu(x)d\nu(y) = \int_G f(x)d(\mu * \nu)(x)$, where the unit is the Dirac measure at the identity of G, δ_e. With the notation $\check{f}(x) = \overline{f(x^{-1})}$, which we shall be using from now on, the involution is given by $\mu^*(f) = \overline{\mu(\check{f})}$ [22].

A representation U of G on H is also a representation of $M^1(G)$ and is written

$$\mu \mapsto U(\mu) = \int_G d\mu(x)\, U(x),$$

whose restriction to $L^1(G)$ is non-degenerate,

$$f \mapsto U(f) = \int_G dx\, f(x)U(x). \tag{50}$$

There is, in fact, a bijective correspondence between the unitary irreducible representations of G and the non-degenerate representations of $L^1(G)$. In particular, to the left-regular representation of G corresponds the operator

$$L(f) \equiv \hat{f} = \int_G dx f(x)L(x), \tag{51}$$

whose restriction to $\mathcal{N}_{\varphi_a} = L^2(G) \cap L^\infty(G)$ is just that derived earlier as the Fourier representation of $\mathbb{K}^a(G)$ and denoted $\lambda(f)$. Furthermore, when restricted to the space $L^1(G) \cap L^2(G)$, $L(f)$ acts by convolution:

$$L(f)g = f * g, \qquad g \in L^1(G) \cap L^2(G). \tag{52}$$

$L(f)$ also satisfies:

$$\begin{aligned}
L(f * g) &= \hat{f} \cdot \hat{g}; \\
L(f^*) &= \int_G dx\, \triangle_G x^{-1} \overline{f(x^{-1})} L(x) = \int_G dx\, \overline{f(x)} L^\dagger(x) \\
&= \hat{f}^\dagger.
\end{aligned}$$

The operators which constitute the image of λ form the von Neumann algebra $\widehat{L^\infty(G)}$.

The Abelian C*-algebra $C_o(G)$ of complex functions vanishing at infinity, with norm $\|f\| = sup|f(x)|$ and involution given by the complex conjugation, has as its dual the algebra $M^1(G)$, the duality pairing being given by

$$\mu(f) = \langle \mu, f \rangle = \int_G d\mu(x) f(x) = \int_G dx\ g(x) f(x),$$

if $d\mu(x) = g(x)dx$. The same duality relation holds between the pair $L^\infty \supset C_o(G)$ and $L^1 \subset M^1(G)$ as a linear functional on the latter, since $L^1(G)^* = L^\infty(G)$. While we have $L^\infty(G)_* = L^1(G)$, the dual of L^∞ is not L^1, but just contains it [15].

4.2 The Symmetric Kac Algebra of a Group

The von Neumann algebra $\widehat{L^\infty(G)}$, generated by left-regular representations of a l.c. group G, is denoted $\mathcal{M}(G)$. Their generators $L(x),\ x \in G$, act on $L^2(G)$ by

$$[L(x)f](y) = f(x^{-1}y).$$

The norm is given by $\|L(x)\| = sup\{\|L(x)f\|,\ \|f\|_2 = 1\}$ and the involution by hermitian conjugation. The product in $\mathcal{M}(G)$ is the representation of the group product, $L(x)L(y) = L(xy)$, with unit $\mathbf{1} = L(e) = I$. Every element (operator) in $\mathcal{M}(G)$ is a linear combination of all generators, with functions in $L^1(G)$ as coefficients,

$$\hat{f} = \int_G dx\ f(x) L(x), \qquad f \in L^1(G), \tag{53}$$

just the image of the Fourier representation of $\mathbb{K}^a(G)$ given in (45). These operators act on $L^2(G)$ by

$$[\hat{f}g](x) = \int_G dy\ f(y)[L(y)g](x) = \int_G dy\ f(y) g(y^{-1}x). \tag{54}$$

If we further restrict to $g \in L^2(G) \cap L^1(G)$, (54) turns out to be just the convolution (52). The product of operators is written as

$$\begin{aligned} \hat{f} \cdot \hat{g} &= \int_G dx \int_G dy\ f(x) g(y) L(xy) = \int_G dz \int_G dx\ f(x) g(x^{-1}z) L(z) \\ &= \int_G dz\ (f * g)(z) L(z). \end{aligned} \tag{55}$$

With the operator product (55), $\mathbb{K}^s(G) = (\mathcal{M}(G), \triangle, \kappa, \varphi_s)$ is the *symmetric Kac algebra* of the group G. The other operations are

$$\triangle L(x) = L(x) \otimes L(x); \tag{56a}$$

$$\kappa(L(x)) = L(x^{-1}) = L^{-1}(x) = L^\dagger(x); \tag{56b}$$

$$\varphi_s(a) = \begin{cases} \|f\|_2^2 & \text{if } a = \hat{f}^\dagger \cdot \hat{f} \\ +\infty & \text{otherwise.} \end{cases} \qquad a \in \hat{M}^+ \tag{56c}$$

Just for completeness and better visualization of the structure, we rewrite the above expressions in terms of the linear combinations (53), whose product has been given in (55):

$$\triangle(\hat{f}) = \int_G dx\, f(x)\, L(x) \otimes L(x); \tag{57a}$$

$$\kappa(\hat{f}) = \int_G dx\, f(x)\, L(x^{-1}) = \int_G dx\, \triangle x^{-1} f(x^{-1})\, L(x); \tag{57b}$$

$$\varphi_s(\hat{f}) = f(e), \qquad f \in C(G) * C(G). \tag{57c}$$

If $F \in C(G \times G)$, we have

$$\hat{W} F(x,y) = F(y^{-1}x, y), \qquad \hat{W}^* F(x,y) = F(yx, y), \tag{58a}$$

$$Jf(x) = \overline{f(x)}, \;\; f \in C(G). \tag{58b}$$

In order to see how $\hat{W}$ generates λ, let us consider the space $L^2(G, L^2(G))$ of L^2-valued functions on G. It turns out to be isomorphic to $L^2(G) \otimes L^2(G)$ by the association $\phi(y)(x) = F(x,y)$, where $\phi(y) \in L^2(G)$. Actually, λ is generated by the left-regular representation L of G, whose action on $\phi(y)$ can be written, with the help of (58), as

$$\begin{aligned} [L(y)\phi(y)](x) = \phi(y)(y^{-1}x) &= F(y^{-1}x, y) \\ &= [\hat{W}F](x,y). \end{aligned}$$

This shows that the generator L, as a bounded function from G to $\mathcal{B}(L^2(G))$, can be seen as the operator $\hat{W} \in \mathcal{B}(L^2(G)) \otimes L^\infty(G)$.

The modular operator on $\mathbb{K}^s(G)$ is given by the Radon-Nikodým derivative $\triangle_{\varphi_s} = \frac{d\varphi_a}{d(\varphi_a \circ \kappa)}$ of the trace $\varphi_a = \mu_l$ on $\mathbb{K}^a(G)$. Since by a quick calculation one obtains $\mu_l \circ \kappa = \mu_r$, it turns out from the definition of the modular function (48) that the modular operator is just $\triangle_G$. The

modular function acts on $L^2(G)$ by pointwise multiplication and, for $f \in H_{\varphi_s}$,

$$
\begin{aligned}
[\sigma_t^{\varphi_s}(L(x))f](y) &= [\triangle_G^{it} L(x) \triangle_G^{-it} f](y) \\
&= (\triangle_G y)^{it} [L(x) \triangle_G^{-it} f](y) \\
&= (\triangle_G y)^{it} (\triangle_G (x^{-1}y))^{-it} f(x^{-1}y) \\
&= (\triangle_G x)^{it} f(x^{-1}y),
\end{aligned} \tag{59}
$$

which gives $\sigma_t^{\varphi_s}(L(x)) = (\triangle_G x)^{it} L(x)$.

As the base space for the dual of $\mathbb{K}^s(G)$, the predual $\mathcal{M}(G)_*$ is the von Neumann algebra of functionals $\hat{\omega}_{f,g} : \mathcal{M}(G) \to \mathbb{C}$ such that $\hat{\omega}_{f,g}(\hat{h}) = (\hat{h}(f)|g)$, which is isomorphic to the *Fourier algebra* $A(G)$ of those functions h which can be written in the form $h = f * \check{g},\ f, g \in L^2(G)$. Their identification comes from (28) and is given through the function $\hat{\omega}_{f,g}(x) \equiv \langle L(x^{-1}), \hat{\omega}_{f,g} \rangle = (f * \check{g})(x)$. The Fourier representation in this case also follows from (29) and (58),

$$
\begin{aligned}
(\hat{W}(f \otimes g)|h \otimes l) &= \int_G dy\, g(y) \overline{l(y)} \int_G dx\, \overline{h(x)} f(y^{-1}x) \\
&= \int_G dy\, g(y) \overline{(h * \check{f})(y) l(y)} \\
&= \int_G dy\, g(y) \overline{\hat{\omega}_{h,f}(y) l(y)},
\end{aligned}
$$

from which we get $\hat{\lambda}(\hat{\omega}_{h,f}) l = \hat{\omega}_{h,f} l$. It turns out that the Fourier representation is the identity. By the Cauchy-Schwarz inequality we obtain that $A(G)$ is contained in $L^\infty(G)$ (its normic closure in this algebra is just $C_o(G)$), and from (39) and the last expression we obtain that it has the usual L^∞-pointwise product as operation.

4.3 Decomposition into Irreducibles

The reducible representations of a type I group G can be decomposed into irreducible representations in a unique way [18]. However, the previous knowledge of the unitary dual $\hat{G}$ of G is necessary to the actual realization of the decomposition. The dual is a space consisting of equivalence classes of unitary irreducible representations with its Mackey-Borel structure and a Plancherel (this name will be justified below) measure associated to the Haar measure on G [7]. The Plancherel measure and the type I Mackey-Borel structure will allow us to sum (or integrate) on $\hat{G}$ without ambiguities [18, 20]. We proceed to obtain the decomposition in

this section. We will take the regular representation case as a starting point and arrive at the decomposition of the von Neumann algebra it generates and of the Hilbert spaces they act upon (see, for example, (8)). For the left-regular representations, the decomposition can be written in the form

$$L = \int_{\hat{G}}^{\oplus} d\mu(\alpha)\, T_\alpha,$$

where $\alpha \in \hat{G}$ and $d\mu(\alpha)$ is a Plancherel measure. This corresponds to the direct integral decomposition $\mathcal{M}(G) = \int_{\hat{G}}^{\oplus} d\mu(\alpha)\, \mathcal{M}_\alpha(G)$ of the von Neumann algebra underlying $\mathbb{K}^s(G)$. Since the operators $T_\alpha(x)$ provide irreducible representations of G, there should be an analogous decomposition of the representation of $L^1(G)$,

$$L(f) = \int_{\hat{G}} d\mu(\alpha)\ T_\alpha(f), \tag{60}$$

with each summand given by

$$T_\alpha(f) \equiv \hat{f}_\alpha = \int_G dx\ f(x) T_\alpha(x). \tag{61}$$

This gives a new aspect to formula (53),

$$L(f) = \int_{\hat{G}} d\mu(\alpha) \int_G dx\ f(x) T_\alpha(x). \tag{62}$$

Formula (61) is the generalized Fourier transform of $f \in L^1(G)$, whose outcome is the operator-valued function $\hat{f}_\alpha$ on $\hat{G}$. Its image belongs to the von Neumann algebra $\mathcal{M}_\alpha(G)$, which acts on the Hilbert space H_α such that $L^2(G) = \int_{\hat{G}}^{\oplus} d\mu(\alpha)\, H_\alpha$.

As regards the weight φ_s, it is a trace if and only if G, or $\mathbb{K}^s(G)$, is unimodular. We can easily show it using (57c) and (55). Restricting to $f \in L^1(G) \cap L^2(G)$, we get

$$\varphi_s(\hat{f} \cdot \hat{f}^\dagger) = (f * f^*)(e) = \int_G dx \bigtriangleup_G x\, |f(x)|^2 \tag{63a}$$

$$\varphi_s(\hat{f}^\dagger \cdot \hat{f}) = (f^* * f)(e) = \int_G dx\, |f(x)|^2, \tag{63b}$$

where we have also used (49) to obtain (63b). In the unimodular case we have the Plancherel formula, which involves a well-defined decomposition for $\varphi_s = \mathrm{Tr}$, as explained in Ref. [7]. In the general case, where symmetric Kac algebras of a non-unimodular type I group are considered, we can suppose also the weight φ_s to be decomposed according to

$$\varphi_s = \int_{\hat{G}}^{\oplus} d\mu(\alpha)\, \varphi_\alpha. \tag{64}$$

It will be sometimes useful to extend abusively the domain of φ_s to the generators $L(x)$, which can be regarded as left-regular representations of the Dirac measures $\delta_x \in M^1(G)$: $L(x) = \int_G dy\, \delta_x(y) L(y)$. In this sense we write

$$\delta_e(x) = \varphi_s(L(x)) = \int_{\hat{G}} d\mu(\alpha)\, \varphi_\alpha(T_\alpha(x)), \tag{65}$$

which is to be regarded as the explicit general expression for the Dirac delta distribution on the group.

Going further, from (63a) we can write, for $f \in L^1(G) \cap L^2(G)$, two expressions: on one hand, $\varphi_s(\hat{f}^\dagger \cdot \hat{f}) = \int_G dx\, |f(x)|^2$; on the other hand, $\varphi_s(\hat{f}^\dagger \cdot \hat{f}) = \int_{\hat{G}} d\mu(\alpha)\, \varphi_\alpha[(\hat{f}^\dagger \cdot \hat{f})_\alpha]$. We obtain, consequently,

$$\int_G dx\, |f(x)|^2 = \int_{\hat{G}} d\mu(\alpha)\, \varphi_\alpha[(\hat{f}^\dagger \cdot \hat{f})_\alpha]$$

as a generalization of the Plancherel formula, where

$$(\hat{f}^\dagger \cdot \hat{f})_\alpha = \hat{f}^\dagger_\alpha \cdot \hat{f}_\alpha = \int_G dx\, (f^* * f)(x) T_\alpha(x).$$

Since $f \in L^2(G)$, it follows that, for *almost all* α, $\varphi_\alpha[\hat{f}^\dagger_\alpha \cdot \hat{f}_\alpha] < \infty$, and we can conclude that $\hat{f}_\alpha \in \mathcal{N}_{\varphi_\alpha}$ for *almost all* α. Here and in the following *almost all* α includes that set of representations whose complement in the unitary dual has zero Plancherel measure, that is, the support of this measure. It is generally identified with the set of higher dimensional representations. For example, in the half-plane canonical group case they are just the infinite-dimensional $T_\pm$. From (64) we also have that, for almost all α, the φ_α are normal, faithful and semi-finite weights on $\mathcal{M}_\alpha(G)$.

With the above weight decomposition we are able to write out the inverse of the generalized Fourier transform (61). From (57c) it follows that

$$f(x) = \varphi_s[L^\dagger(x)\hat{f}],$$

whose decomposition

$$f(x) = \int_{\hat{G}} d\mu(\alpha)\, \varphi_\alpha[T^\dagger_\alpha(x)\hat{f}_\alpha] \tag{66}$$

gives $f(x)$ in terms of the operator-valued function $\hat{f}_\alpha$ on $\hat{G}$. Writing

$$f_\alpha(x) \equiv \varphi_\alpha[T^\dagger_\alpha(x)\hat{f}_\alpha] \tag{67}$$

and recalling that $f \in L^1(G)$, it follows from (66) that $\int_G dx\,|f_\alpha(x)| < \infty$ for almost all α, that is, that $f_\alpha \in L^1(G)$ for almost all α.

Notice that the generalized Fourier transform defined in (61) and (66) is faithful as a map between G and its dual if and only if the Plancherel measure accounts for every element of $\hat{G}$. Since the Plancherel measure is concentrated on the highest dimensional representations, it may happen in some cases, like those of the Heisenberg [10, 25] and the special half-plane [11] groups, that there are irreducible inequivalent representations on $\hat{G}$ which are missed in formulas (60) and (66).

We have up to now collected the decompositions of $\mathcal{M}(G)$, of its generators $L(x)$, of the Hilbert space $L^2(G)$ and of the Haar weight φ_s. The question coming naturally to the mind is whether or not these (irreducible) components constitute a Kac algebra. The answer is negative, because the components φ_α of φ_s are not Haar weights in the Hopf-von Neumann algebra $\mathbb{H}_\alpha(G)$ generated by $T_\alpha(x)$. The structure of $\mathbb{H}_\alpha(G)$ for fixed $\alpha \in \hat{G}$ comes straight from the decomposition of L:

$$T_\alpha(x)T_\alpha(y) = T_\alpha(xy) \tag{68a}$$

$$I = T_\alpha(e) \tag{68b}$$

$$\triangle_\alpha(T_\alpha(x)) = T_\alpha(x) \otimes T_\alpha(x) \tag{68c}$$

$$\kappa_\alpha(T_\alpha(x)) = T_\alpha^\dagger(x). \tag{68d}$$

If we take φ_α and try to verify the second axiom for Haar weights, for example, we get from the two sides of (32):

$$\begin{aligned}
(id \otimes \varphi_\alpha)[(I \otimes T_\alpha^\dagger(y))\triangle_\alpha(T_\alpha(x))] &= \varphi_\alpha(T_\alpha(y^{-1}x))\,T_\alpha(x),\\
\kappa_\alpha(id \otimes \varphi_\alpha)[\triangle_\alpha(T_\alpha^\dagger(y))(I \otimes T_\alpha(x))] &= \varphi_\alpha(T_\alpha(y^{-1}x))\,T_\alpha(y),
\end{aligned}$$

which implies that axiom if and only if $x = y$. Since there is no warrant that $\varphi_\alpha(T_\alpha(y^{-1}x))$ would have as outcome $x = y$ (we have instead (65)), we conclude that φ_α is not Haar. Conversely, by the Haar weight axioms, it can be proved that a weight φ' is a Haar weight on $\mathbb{H}_\alpha(G)$ if and only if $\varphi'(T_\alpha(x)) = \delta_e(x)$.

The elements of $\mathbb{H}_\alpha(G)$ are written

$$\tau_\alpha(f) \equiv \hat{f}_\alpha = \int_G dx\, f(x)\, T_\alpha(x), \qquad f \in L^1(G). \tag{69}$$

This means that they are the images of the inequivalent irreducible representations of $L^1(G)$ corresponding to the T_α representations of G. In analogy with the relation between the representation L and λ, where the latter is a restriction of L to $\mathcal{N}_{\varphi_a} = L^\infty(G) \cap L^2(G)$, formula (69) is regarded as a restriction of formula (61) to the respective decomposition of $\mathcal{N}_{\varphi_a}$, that is, as an α-component τ_α of the Fourier representation λ. It is thus natural to look for its generator. We shall, in what follows, restrict ourselves to separable and type I semi-direct product groups of the type $G = S \oslash N$, where N is an Abelian normal subgroup and S is a unimodular group. This restriction on the group will provide more explicit formulas for the Kac algebra decomposition, while retaining enough generality to allow the consideration not only of the half-plane example envisaged here but also of other cases of physical interest, like the Euclidean motion group $E(2)$, the correct canonical group for the phase space of the circle [13]. Elements of G will be denoted by $x = (s, n),\ y = (r, l)$, etc., the identity by $(e, 0)$ and the product by $(s, n)(r, l) = (sr, n + \phi_s(l))$, where ϕ_s is a homomorphism on N, the action of S. In this case, a generalization of what was presented in section 3 regarding the group $\mathbb{R}_+ \oslash \mathbb{R}$ is provided by Mackey's theory of induced representations applied to semi-direct product groups (see also Refs. [23, 11, 4]). If V_y are irreducible representations (characters) of N, labeled by $y \in \hat{N}$, that theory says that the Hilbert space H_y is formed by those functions f_y which satisfy:

- the map $(s, n) \in G \mapsto f_y(s, n) \in \mathbb{C}$ is measurable;
- $f_y((s, n)(e, l)) = V_y^{-1}(l)\, f_y(s, n)$;
- $\int_{G/N} d\mu(s)\, |f_y(s, n)|^2 < \infty$,

where $d\mu(s)$ is a G-invariant measure on $G/N \sim S$ (notice that H_y differs from H_α in that the label α represents classes of inequivalent representations while y represents all (irreducible) representations). The action of $(s, n) \in G$ on $r \in S$, denoted, $(s, n) \cdot r$, is defined by taking the S-component of the product $(s, n)(r, 0) = (sr, n) = (sr, 0)(e, \phi_{sr}^{-1}(n))$, according to the decomposition $(s, n) = (s, 0)(e, \phi_s^{-1}(n))$ of G in terms of its parts S and N, that is, $(s, n) \cdot r \equiv sr \in S$. By the same decomposition, the second condition implies that the functions f_y can be written as

$$f_y(s, n) = V_y^{-1}(\phi_s^{-1}(n))\xi(s), \qquad f_y(s, 0) \equiv \xi(s) \in L^2(S). \tag{70}$$

Actually, (70) expresses an isomorphism between H_y (H_α) and $L^2(S)$ for each label $y\,(\alpha)$. In what follows we will suppose that the irreducible representations have been already classified, that is, we will work on H_α. On these spaces the irreducible unitary induced representations T_α of G by V_α are given by

$$[T_\alpha(x)\xi](s) = V_\alpha(\sigma(x^{-1}\cdot s; x))\xi(x^{-1}\cdot s), \tag{71}$$

where $\sigma(r;x)$ is a "gaugefied" cocycle on G, or a (S,G)-cocycle relative to the invariant class of $d\mu(s)$ [27, 11], that is, a Borel map $\sigma : S\times G \to N$ which satisfies

$$\sigma(r;e) = 0; \tag{72}$$

$$\sigma(y\cdot r;x) - \sigma(r;xy) + \sigma(r;y) = 0. \tag{73}$$

It is given explicitly by $\sigma(r;s,n) = \phi_{sr}^{-1}(n)$. "Gaugefied" cocycles appear already in usual Quantum Mechanics, even in its discretized version, in which the Euclidean phase space is replaced by $\mathbb{Z}_n \otimes \mathbb{Z}_n$ [1].

Formula (69) will have an important role in our work. It generalizes Weyl's formula [28] in the sense that it associates (L^1) functions on the group to irreducible operators on the subgroup S. In order to find explicitly the generator of the representation $T_\alpha(\tau_\alpha)$ of $L^1(G)$, we consider functions $\psi \in L^2(G,H_\alpha) \sim L^2(G,L^2(S))$ and, putting $\psi(x)(s) = G(s,x),\ x\in G$ we define an isomorphism between the $L^2(S)$-valued functions on G and the space $L^2(S)\otimes L^2(G)$. Now, the induced irreducible representations on $\psi(x)\in L^2(S)$ are given by

$$\begin{aligned} [T_\alpha(x)\psi(x)](s) &= V_\alpha(\sigma(x^{-1}\cdot s;x))\psi(x)(x^{-1}\cdot s) \\ &= V_\alpha(\sigma(x^{-1}\cdot s;x))G(x^{-1}\cdot s,x) \equiv [\hat{W}^\alpha G](s,x). \end{aligned} \tag{74}$$

This shows that the generator of τ_α, the function T_α in $L^\infty(G,\mathcal{B}(L^2(S)))$, can be seen as an operator $\hat{W}^\alpha \in \mathcal{B}(L^2(S))\otimes L^\infty(G)$. Operator $\hat{W}^\alpha$ is analogous to $\hat{W}$ not only in what regards the generation of Fourier representations but also because it implements the coproduct (68c). This can be shown by recalling the definitions of the induced representations T_α on $H_\alpha(G)$ and $L^2(S)$, and comparing the two actions below,

$$\begin{aligned} &[\hat{W}^\alpha(I\otimes T_\alpha(z))\hat{W}^{\alpha *}G_\alpha](s,x) = \\ &\qquad = V_\alpha^{-1}(-\sigma(x^{-1}\cdot s;x))V_\alpha^{-1}(\sigma(x^{-1}\cdot s;z^{-1}x))\,G(z^{-1}\cdot s,z^{-1}x) \\ &\triangle_\alpha T_\alpha(z)G_\alpha(s,x) = V_\alpha^{-1}(\sigma(s;x^{-1}))\,G(z^{-1}\cdot s,z^{-1}x). \end{aligned}$$

The left hand sides above turn out to be equal if we substitute in the right hand sides the identity $\sigma(x^{-1} \cdot s; z^{-1}x) = \sigma(s; z^{-1}) + \sigma(x^{-1} \cdot s; x)$, straightforwardly obtained from (73). We have thus that W^α is the fundamental operator of $\mathbb{H}_\alpha(G)$. It is easily verified that it satisfies the pentagonal relation.

Turning back to (69), we obtain by (71) that the operators $\hat{f}_\alpha$ act on $L^2(S)$ by

$$[\hat{f}_\alpha \xi](r) = \int_G d^l\mu(s,n)\, V_\alpha(\sigma(s^{-1}r; s, n)) f(s,n) \xi(s^{-1}r). \tag{75}$$

Since the right invariant measure on G is the product of the invariant measures on S and N, we have $d^l\mu(s,n) = \triangle(s,n)\, d\mu(s)\, d\mu(n)$. After the change of variables $s^{-1}r = t$ and by Fubini's theorem, (75) reads

$$[\hat{f}_\alpha \xi](r) = \int_S d\mu(t)\, K_f^\alpha(r,t) \xi(t)$$

in terms of the kernel $K_f^\alpha(r,t)$ given by

$$K_f^\alpha(r,t) = \int_N d\mu(n)\, \triangle(rt^{-1}, n)\, V_\alpha(\sigma(t; rt^{-1}, n)) f(rt^{-1}, n).$$

Introducing a kernel will enable us to write out an explicit formula for the weight φ_α. Also the following result will help: *the modular function of a semi-direct product group $G = S \oslash N$ is only a function on S.* This is proved by using $d^r\mu(x^{-1}y) = \triangle_G x d^r\mu(y)$ and the invariance of the measures on S and N:

$$\begin{aligned}
d^r\mu((s,n)^{-1}(r,l)) &= d^r\mu(s^{-1}r, \phi_s^{-1}(l-n)) \\
&= d\mu(s^{-1}r) \wedge d\mu(\phi_s^{-1}(l-n)) \\
&= \left| \frac{\partial \phi_s^{-1}(l)}{\partial l} \right| d\mu(r) \wedge d\mu(l) \\
&= \triangle_G(s,n)\, d^r\mu(r,l),
\end{aligned}$$

that is,

$$\triangle_G(s,n) = \left| \frac{\partial \phi_s^{-1}(l)}{\partial l} \right| \equiv \triangle(s), \tag{76}$$

and, in particular, $\triangle_G(e,n) = \triangle(e) = 1$. In G_{hp} we have $d^r(\lambda, a) = \frac{d\lambda}{\lambda} da$ and the left-invariant measure is easily verified to be $d^l(\lambda, a) = d\lambda da$, which implies that $\triangle_{G_{hp}}(\lambda, a) = \triangle(\lambda) = \lambda$.

Turning back to the general case, a trace can be introduced on $\mathbb{H}_\alpha(G)$ by

$$\begin{aligned} \mathrm{Tr}_\alpha(\hat{f}_\alpha) &= \int_S d\mu(t)\, K_f^\alpha(t,t) && \text{(77a)} \\ &= \int_G d\mu(t)\, d\mu(n)\, V_\alpha(\sigma(t;e,n)) f(e,n). && \text{(77b)} \end{aligned}$$

Formula (77a) is a good trace definition because the kernels satisfy

$$\int_S d\mu(t)\, K_f^\alpha(r,t) K_g^\alpha(t,s) = K_{f*g}^\alpha(r,s),$$

which implies $\mathrm{Tr}_\alpha(\hat{f}_\alpha^* \hat{f}_\alpha) = \mathrm{Tr}_\alpha(\hat{f}_\alpha \hat{f}_\alpha^*)$. We will now introduce an explicit decomposition of the Haar weight in terms of the trace:

$$\begin{aligned} \varphi_\alpha(\hat{f}_\alpha) &\equiv \mathrm{Tr}_\alpha(\triangle \hat{f}_\alpha) = \int_S d\mu(t)\, \triangle(t) K_f^\alpha(t,t) \\ &= \int_G d\mu(t)\, d\mu(n)\, \triangle(t)\, V_\alpha(\sigma(t;e,n)) f(e,n) \\ &= \int_G d^l\mu(t,n)\, V_\alpha(\sigma(t;e,n)) f(e,n), \end{aligned} \tag{78}$$

where $\triangle$ is given by (76). Clearly it is not a trace. For example, in the half-plane group we have

$$\varphi_\pm(\hat{f}_\pm) = \frac{1}{2\pi} \int_{G_{hp}} d\lambda da\, e^{\pm ia\lambda} f(1,a),$$

which is a decomposition of the Haar weight φ_s, since

$$\varphi_s(\hat{f}) = \sum_\pm \varphi_\pm(\hat{f}_\pm) = \frac{1}{\pi} \int_{\mathbb{R}_+ \times \mathbb{R}} d\lambda da\, \cos(a\lambda) f(1,a) = f(1,0).$$

Computing $\varphi_\alpha(T_\alpha(r,l))$, which should be $\delta_e(r,l)$ if φ_α were a Haar weight, the formula above provides another way to see why that does not happen:

$$\varphi_\alpha(T_\alpha(r,l)) = \delta_e(r) \int_S d\mu(t) \triangle(t)\, V_\alpha(\sigma(t;e,l)).$$

For G_{hp} this gives

$$\varphi_\pm(T_\pm(\lambda,a)) = \frac{\delta_1(\lambda)}{2\pi} \int_{\mathbb{R}_+} d\rho\, e^{\pm ia\rho} = \frac{\delta_1(\lambda)}{2\pi} \left(\pi\delta(a) \pm \frac{i}{a} \right),$$

while $\sum_\pm \varphi_\pm(T_\pm(\lambda,a)) = \delta_1(\lambda)\delta(a)$. Furthermore, from formula (78) the function (67) reads explicitly

$$f_\alpha(r,l) = \int_G d^l\mu(t,n)\, V_\alpha(\sigma(t;e,n)) f((r,l)(e,n)), \qquad f \in L^1(G).$$

We turn now our attention to the predual of $\mathcal{M}_\alpha(G)$ which, in analogy with the Fourier algebra, we call $A_\alpha(G)$. As in that case, we introduce the representative function $\hat{\omega}^\alpha_{\xi,\chi}(x)$ of this new algebra by

$$\hat{\omega}^\alpha_{\chi,\xi}(x) \equiv \langle \hat{\omega}^\alpha_{\chi,\xi}, T^\dagger_\alpha(x) \rangle,$$

which, by definition of $\hat{\omega}^\alpha_{\xi,\chi}$ as a linear form, is given by the scalar product

$$\begin{aligned} \hat{\omega}^\alpha_{\chi,\xi}(s,n) &= (T^\dagger_\alpha(s,n)\chi|\xi)_{L^2(S)} = (\chi|T_\alpha(s,n)\xi)_{L^2(S)} \\ &= \int_S d\mu(t)\, V^{-1}_\alpha(\sigma(s^{-1}t;s,n))\chi(t)\check{\xi}(t^{-1}s), \end{aligned} \tag{79}$$

where, we recall, $\check{\xi}(s) = \overline{\xi(s^{-1})}$. The product on $A_\alpha(G)$ is obtained by the duality shown in (36) from the coproduct on $\mathbb{H}_\alpha(G)$ and is the same Abelian pointwise product of $A(G)$, since (68c) is symmetric, and of the same kind of the coproduct on $K^s(G)$. Another point, the involution on $A_\alpha(G)$ is straightforwardly seen from (37) to be just the complex conjugation. These facts show that $A_\alpha(G)$ is very similar to $A(G)$, differing from it only in that their elements should be written according to (79) and depend on the labels $\alpha \in \hat{G}$. As $A(G)$ is contained in $L^\infty(G)$, this suggests that $A_\alpha(G)$ be contained in some space alike. To see it better, we compute the modulus of $\hat{\omega}^\alpha_{\chi,\xi}(s,n)$ and, using the Cauchy-Schwarz inequality, obtain

$$\begin{aligned} |\hat{\omega}^\alpha_{\chi,\xi}(s,n)| &= |(\chi|T_\alpha(s,n)\xi)| \\ &\leq \|\chi\|_2\|T_\alpha(s,n)\xi\|_2 \\ &= \|\chi\|_2\|\xi\|_2 < \infty, \end{aligned}$$

since T_α is unitary and $\chi, \xi \in L^2(S)$. Thus, $|\hat{\omega}^\alpha_{\chi,\xi}(s,n)|$ is essentially bounded and we can say that $A_\alpha(G)$ is contained in $L^\infty(G)$. The index α just indicates the dependence on the labels in $\hat{G}$.

Using the explicit form of the generator $\hat{W}^\alpha$ we can determine the representation $\hat{\tau}_\alpha$ of $\mathcal{M}_\alpha(G)_* = A_\alpha(G)$ by formula (29),

$$(\hat{W}^\alpha(\xi,f)|\chi \otimes g)_{L^2(S)\otimes L^2(G)} = (f|\hat{\tau}_\alpha(\hat{\omega}^\alpha_{\chi,\xi})g)_{L^2(G)}, \tag{80}$$

where $\xi, \chi \in L^2(S)$ and $f, g \in L^2(G)$. The left hand side gives

$$\begin{aligned} (\hat{W}^\alpha(\xi,f)|\chi \otimes g) &= \int_G d\mu^l(s,n)\, f(s,n) \int_S d\mu(t)\, V_\alpha(\sigma(s^{-1}t;s,n))\overline{\chi(t)}\xi(s^{-1}t)\, \overline{g(s,n)} \\ &= \int_G d\mu^l(s,n)\, f(s,n)\, \overline{\hat{\omega}^\alpha_{\chi,\xi}(s,n) g(s,n)}. \end{aligned}$$

Comparison with the right hand side of (80) yields $\hat{\tau}_\alpha = id$. Taking $\hat{\tau}_\alpha$ as an α-component of $\hat{\lambda}$ and recalling that the dual of $K^s(G)$ is built on the image of $\hat{\lambda}$, we conclude that the dual of $\mathbb{H}_\alpha(G)$ is built on $\hat{\tau}_\alpha(A_\alpha(G)) \subset L^\infty(G)$, that is, the dual is contained in $\mathbb{H}^a(G)$. We also obtain from (39) that, while $\mathcal{M}_\alpha(G)$ acts on $L^2(S)$, $A_\alpha(G)$ acts on $L^2(G)$ (by the pointwise product), which explains the asymmetry of the double scalar product in (80). The dual version of that formula,

$$(W^\alpha(f,\xi)|g \otimes \chi)_{L^2(G)\otimes L^2(S)} = (\xi|\tau_\alpha(\omega_{g,f})\chi)_{L^2(S)}, \tag{81}$$

which is also asymmetric, involves the representation (69) and the operator $W^\alpha = \sigma \circ \hat{W}^{\alpha *} \circ \sigma$ $(f \in L^2(G),\ \xi \in L^2(S))$,

$$[W^\alpha(f,\xi)](s,n;r) = V_\alpha^{-1}(\sigma(r;s,n))\, f(s,n)\xi(sr).$$

The left hand side of (81) then gives

$$\begin{aligned}(W^\alpha(f,\xi)|g\otimes\chi) &= \int_S d\mu(t)\,\xi(t)\int_G d\mu^l(s,n)\,V_\alpha^{-1}(\sigma(s^{-1}t;s,n))\overline{g(s,n)}f(s,n)\,\overline{\chi(s^{-1}t)} \\ &= \int_S d\mu(t)\,\xi(t)\overline{\int_G d\mu^l(s,n)\,\omega_{g,f}(s,n)[T_\alpha(s,n)\chi](t)},\end{aligned}$$

where the first equality involves a change of variables in S and we have identified $\omega_{g,f} = g\overline{f}$ by (28). Comparison of this result with the right hand side of (81) corroborates formula (69) for τ_α. If we also introduce in $A_\alpha(G)$ the α-component of the coproduct in $\mathbb{H}^a(G)$, this will be implemented by W^α, in the same way that $\hat{W}^\alpha$ implements (68c).

5 Quantization on the Half-Plane

We have now at hand a powerful structure to describe quantization. A generalization of the Weyl-Wigner correspondence prescription is incorporated in Kac (group) duality. Our objective in this section is to specialize to the half-plane case the last section results, particularly those concerning the decomposition of the Kac algebras. We shall show that the Hopf-von Neumann algebra generated by the irreducible operators, together with its dual, does provide the framework in which quantization can be described as an irreducible component in Kac duality theory.

Starting from the group G_{hp} as the closest algebraic entity associated to the half-plane phase space, we necessarily have to consider – if we are thinking about duality – the two Kac algebras $\mathbb{K}^s(G_{hp})$ and $\mathbb{K}^a(G_{hp})$. The decomposition of the first according to the dual space $\widehat{G_{hp}}$ leads to the family of Hopf-von Neumann algebras $\mathbb{H}_\alpha(G_{hp})$, which inherit most of their structure from the Kac algebras they come from. Though group duality is lost at the Hopf-von Neumann level, a well-defined formula for the decomposition of the Fourier representation persists. Adaptation of formula (69) and its inverse (66), although not representing a bijection between the group and its dual, provides a well-defined mapping between functions and irreducible operators.

The cohomological differences between the complete-plane and the half-plane cases come from the necessity of central-extending the special canonical group of the former to the Heisenberg group in order to provide a faithful momentum map between its associated Lie and Poisson algebras [13]. Despite these differences, we can regard the Weyl formula as being formula (69) for a fixed value of the label α in terms of the Planck constant, $\alpha = \alpha(h)$. This is easily confirmed if we recall that the unitary dual of the Heisenberg group $H(3)$ is *almost equal* (in the *almost everywhere* sense) to any of the Ω-projective unitary dual [19] of the bidimensional translation group $\mathbb{R}^2$. By the Stone-von Neumann theorem [25], the former is equal to $(\mathbb{Z} - \{0\}) \cup \mathbb{R}^2$, while the latter is just $\mathbb{Z} - \{0\}$. We have shown in a separate paper [2] that the Weyl-Wigner formalism can be described in terms of duality of *projective Kac algebras*. In such a projective duality framework for $\mathbb{R}^2$, Weyl's formula comes from an expression analogous to (69) for the decomposition of the respective Fourier representation, namely

$$\hat{f}_\nu = \int_{\mathbb{R}^2} dxdy\, f(x,y)\, e^{-i\nu(y\hat{q}+x\hat{p})}, \qquad \nu \in \mathbb{Z} - \{0\}, \tag{82}$$

where $\hat{q}$, $\hat{p}$ are the usual position and momentum operators. Comparing (82) with Weyl's formula, we get immediately $\nu = \hbar^{-1}$. Actually, the only formal difference between (69) and the original Weyl's formula, or (82), is that the latter is written in terms of unitary irreducible *projective* operators instead of the linear ones which appear in formula (69). This is a consequence of the necessity of a central extension in the complete-plane case. That is, Quantum Mechanics is a theory on a particular Hilbert space and its operators generate a particular Hopf-von Neumann algebra whose label in the Kac algebra decomposition is just a point in the support of the Plancherel measure on the unitary dual space of the group involved. In the half-plane, as observed at the end of section 2, there is no need for a central extension, since the

cohomology group $H^2(G_{hp}, \mathbb{R})$ is trivial and consequently projective and linear representations are cohomologically equivalent. This enables us to use the simpler, linear representations. Thus, the analogue of Weyl's correspondence formula for the half-plane group is given by (69) for a fixed value of the labels $\pm$. From the dual $\widehat{G_{hp}} = \{+\} \cup \{-\} \cup \mathbb{R}$ we have that (69) is in this case given by

$$\hat{f}_\pm = \int_{G_{hp}} d\lambda da\, f(\lambda, a)\, T_\pm(\lambda, a) \tag{83a}$$

$$f(y) = \int_{G_{hp}} d\lambda da\, f(\lambda, a)\, \chi_y(\lambda), \qquad y \in \mathbb{R}. \tag{83b}$$

Notice that the Hopf-von Neumann algebras $\mathbb{H}_\pm(G_{hp})$ of operators (83a) are quite different from those of functions (83b) and denoted $\mathbb{H}_y(G_{hp})$. They are Abelian for each y and their direct integral over $\mathbb{R}$ can be identified with $\mathbb{H}^a(\mathbb{R})$.

Recalling that the labels $\pm$ correspond to an uncountable infinity of equivalent representations in the support of the Plancherel measure, and taking into account the physical dimensions of the elements of G_{hp} ($[\lambda]$ = length, $[a]$ = momentum, $[\hbar] = [a\lambda] = [\hat{\pi}]$ = action), we take $\pm\hbar^{-1}$ for the representatives of each class instead of $\pm$, and fix the value of the label to be $+\hbar^{-1}$. The quantization map is then given by (we will write $\hbar$ instead of $+\hbar^{-1}$ when it appears only as an indicative of class)

$$\hat{f}_\hbar = \int_{G_{hp}} d\lambda da\, f(\lambda, a)\, e^{\frac{i}{\hbar}a\hat{\rho}} e^{-\frac{i}{\hbar}\ln(\lambda/\lambda_o)\hat{\pi}}, \tag{84}$$

where λ_o is a constant with dimension of length. The self-adjoint operators $\hat{\rho}$ and $\hat{\pi}$ act on the subspace of $L^2(\mathbb{R}_+)$-functions vanishing at 0 and ∞ by

$$\begin{aligned} \hat{\rho}\xi(\rho) &= \rho\xi(\rho) \\ \hat{\pi}\xi(\rho) &= -i\hbar\rho\frac{\partial\xi(\rho)}{\partial\rho}, \end{aligned}$$

and satisfy the commutation relation

$$[\hat{\rho}, \hat{\pi}] = i\hbar\hat{\rho}.$$

The function $f(\lambda, a)$ is recovered from the operator $\hat{f}_\hbar$ by the inverse mapping (66)

$$f(\lambda, a) = \sum_{\pm\hbar^{-1}} \varphi_{\pm\hbar}[T^\dagger_{\pm\hbar}(\lambda, a)\hat{f}_{\pm\hbar}] = \sum_{\pm\hbar^{-1}} f_{\pm\hbar}(\lambda, a), \tag{85}$$

where

$$\varphi_{\pm\hbar}[\hat{f}_{\pm\hbar}] = \frac{1}{2\pi\hbar}\int_{G_{hp}} d\rho db\, e^{\pm\frac{i}{\hbar}b\rho} f(1,b), \tag{86}$$

which gives

$$f_{\pm\hbar}(\lambda,a) = \frac{1}{2\pi\hbar}\int_{G_{hp}} d\rho db\, e^{\pm\frac{i}{\hbar}b\rho} f((\lambda,a)(1,b)). \tag{87}$$

Eq. (85) explicits the fact that the classical L^1-function f has contributions from *almost all* irreducible representations, while Eq. (87) is just the projection of that function into one of its "components".

The symmetric but non-Abelian Hopf-von Neumann algebra $\mathbb{H}_\hbar(G_{hp})$ generated by the irreducible operators $T_\hbar(\lambda,a)$ is then the operator algebra of Quantum Mechanics on the half-plane. Its trivial structure is analogue to that given in (68). On this algebra there is also defined a weight given by the plus sign in (86), which is an irreducible component of the Haar weight on $\mathbb{K}^s(G_{hp})$, as shown in the previous section.

On the dual Abelian Hopf-von Neumann algebra $\mathbb{H}^a(G_{hp})$, a typical $A_\hbar(G_{hp})$-function is that given by

$$\begin{aligned}\hat{\omega}^\hbar_{\chi,\xi}(\lambda,a) &= (\chi|T_\hbar(\lambda,a)\xi)_{L^2(\mathbb{R}_+)} \\ &= \int_{\mathbb{R}_+}\frac{d\rho}{\rho}\; e^{-\frac{i}{\hbar}a\rho}\chi(\rho)\overline{\xi(\rho/\lambda)}.\end{aligned}$$

If we put $\chi=\xi$, $\hat{\omega}^\hbar_{\xi,\xi}\equiv W^\hbar_\xi$ is to be interpreted as a generalization of the Wigner distribution function for the half-plane associated to the state ξ. This is justified, for if we compute the expectation value of the operator $\hat{f}_\hbar$ in the state ξ, it is given by

$$\langle\hat{f}_\hbar\rangle_\xi = (\xi|\hat{f}_\hbar\xi) = \int_{G_{hp}} d\lambda da\, f(\lambda,a)\, W^\hbar_\xi(\lambda,a).$$

This makes clear the role of $W^\hbar_\xi$ as a *quantum probability density*, the same role played by the Wigner distribution in the Euclidean phase space. But notice that things here are quite different from the complete-plane case and this function does not share most of the properties the usual Wigner distribution is known to satisfy. The differences are due to a lack of connection between functions in $A_\hbar(G_{hp})$, like $W^\hbar_\xi$, and L^1-functions, or the respective operator in $\mathcal{M}_\hbar(G_{hp})$. Banach duality is not able to provide an explicit correspondence between these two spaces when the

group is not Abelian self-dual like G_{hp}. In the complete-plane case, $\mathbb{R}^2$ is self-dual and the Banach duality turns out to be just the double Fourier transform on the phase space. Furthermore, the Abelian algebras L^1 and L^∞ over $\mathbb{R}^2$ are isomorphic by the uniqueness of the Fourier transform [22]. This gives rise to the well-known formulas of the Weyl correspondence [10, 25, 12], of which the Wigner distribution function is a particular case corresponding to the density operator. And, since there is no need to consider projective representations in the half-plane case, no 2-cocycle arises, that is, neither the convolution algebra $L^1(G_{hp})$ is twisted nor the pointwise product algebra $L^\infty(G_{hp})$ is deformed by any kind of Moyal-like product.

6 Final Comments

The Weyl-Wigner prescription is based on the Pontryagin duality of Fourier transformations. It calls attention to the central role of duality in quantization, though it only can be expected to hold in the particular case of Abelian canonical groups. We have been concerned with the impact that general Fourier duality can have in quantization. The stage-sets for Fourier analysis are neither groups nor homogeneous spaces, but Kac algebras. General Fourier duality requires actually a pair of algebras and we have considered such a pair of "symmetric" and "Abelian" Kac algebras for a particular, type I but non-Abelian and non-unimodular, canonical group. The decomposition of the first has led to some Hopf-von Neumann algebras on the group, which we have recognized as the natural algebraic arenas where duality plays its role and, furthermore, where we can find out how far it is possible to go with the Weyl-Wigner approach as a guideline to quantize general systems.

The open half-plane which we have examined is perhaps the simplest case presenting some novel, deep features. It is still globally Euclidean – though no more a vector space. Although we have been restricted to a case in which the special canonical group and the phase space manifolds coincide, the group non-triviality requires new algebraic structures. In particular, the group involved being non-unimodular, it is no more a trace which is at work, but its generalization allowing noncommutative integration – a weight. To connect Kac duality and Weyl quantization, we must restrict ourselves to a specific irreducible representation of the group involved. The operators in that representation generate a Hopf-von Neumann algebra which participates in the irreducible decomposition of the symmetric Kac algebra of the canonical group. It is in

general impossible to obtain an explicit correspondence between the L^1- and L^∞-functions on it. From the point of view of Fourier duality, this is standard – Fourier transforms in general map L^1-functions into $\hat{G}$-valued operators (L^∞-functions only if G is Abelian). In the Wigner formalism this corresponds to a failure in the correspondence between the Wigner distribution and its density. Generalized Wigner distribution functions only make sense if related to density operators and, as such, they are defined as the expectation values of the irreducible operators $T_\hbar$. Summing up, generalized Fourier duality in the case treated here does provide a prescription for the quantization of L^1-functions on phase space via a generalized Weyl's formula. Although it is possible to recover the quantizable c-number function from the correspondent Weyl operator, the correspondence is not at all complete, since we cannot relate it to its dual L^∞-function.

The conclusion is that general Fourier duality does provide a firm guide to quantization, though imposing severe restrictions to the simple-minded expectations born from the results concerning those very simple systems for which the phase space is a vector space. Since this duality in only achieved in the Kac algebra framework, we also conclude that, for quantization purposes, algebraic structures beyond groups must be considered.

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Coadjoint Orbits of G_{hp}

For the sake of completeness we shall show here that the half-plane is the unique non-trivial homogeneous symplectic manifold by the action of the group $G_{hp} = \mathbb{R}_+ \oslash \mathbb{R}$. To do that, we realize the group as a 2×2 matrix group by the correspondence

$$(\lambda, a) \mapsto \begin{pmatrix} \lambda^{-1/2} & a\lambda^{1/2} \\ 0 & \lambda^{1/2} \end{pmatrix}.$$

The Lie algebra $\mathcal{G}_{hp}$ can be accordingly realized if we define its generators by

$$L = \frac{d}{dt}(e^t, 0)|_{t=0} = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad A = \frac{d}{dt}(1, t)|_{t=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

A quick computation is enough to verify that L, A realize the algebra $\mathcal{G}_{hp}$,

$$[A, L] = A.$$

To get the adjoint action of G_{hp} on $\mathcal{G}_{hp}$, we write an arbitrary element X of the algebra as $X = X^A A + X^L L$, and compute

$$Ad_{(\lambda,a)}X = (\lambda, a)X(\lambda, a)^{-1} = (aX^L + \lambda^{-1}X^A)A + X^L L.$$

Now, to obtain the coadjoint action of G_{hp} on $\mathcal{G}^*_{hp}$, we first find a dual basis to $X_\nu = \{A, L\}$ in $\mathcal{G}^*_{hp}$ through the duality pairing $\langle \theta^\mu, X_\nu \rangle = \mathrm{Tr}(\theta^\mu X_\nu) = \delta^\mu_\nu$, and get

$$\theta^L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \theta^A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Computing the coadjoint action on an element $\eta = \eta_L \theta^L + \eta_A \theta^A \in \mathcal{G}^*_{hp}, \eta_\nu \in \mathbb{R}$, we find

$$\begin{aligned}
[Ad^*_{(\lambda,a)}\eta](X) &\equiv \langle \eta, Ad^{-1}_{(\lambda,a)}X \rangle \\
&= \langle \eta, X^L L + \lambda(X^A - aX^L)A \rangle \\
&= X^L \eta_L + \lambda(X^A - aX^L)\eta_A.
\end{aligned}$$

To obtain it for any $X \in \mathcal{G}_{hp}$, we compare with

$$[Ad^*_{(\lambda,a)}\eta](X) = (Ad^*_{(\lambda,a)}\eta)_L X^L + (Ad^*_{(\lambda,a)}\eta)_A X^A,$$

which gives finally

$$Ad^*_{(\lambda,a)}\eta = \lambda \eta_A \theta^A + (\eta_L - a\lambda\eta_A)\theta^L. \tag{88}$$

The orbits of this action on $\mathcal{G}^*_{hp}$ are given for all $(\lambda, a) \in G_{hp}$. Analyzing the coefficients of θ^A and θ^L in (88), we conclude that there are basically two kind of orbits: those for which $\eta_A \neq 0$; and those for which $\eta_A = 0$. In the first case the coefficient of θ^A is never zero but that

of θ^L can assume any value in $\mathbb{R}$. This characterizes two half-planes, one for $\eta_A > 0$ and the other for $\eta_A < 0$. In the second case, ($\eta_A = 0$), we have $Ad^*_{(\lambda,a)}\eta = \eta_L\theta^L$, which means that these orbits consist of the infinity of isolated points in the line $\eta_A = 0$. This concludes our analysis, showing that G_{hp} has two 2-dimensional orbits diffeomorphic to the half-plane ($\mathcal{G}^*_{hp_\pm}$) and an uncountable infinity of 0-dimensional orbits ($\mathcal{G}^*_{hp_0}$) in $\mathcal{G}^*_{hp}$.

To conclude, we can also compute the Kirillov symplectic form on the orbits passing by η by the formula $\omega_\eta = \frac{1}{2}C^\sigma_{\mu\nu}\eta_\sigma\theta^\mu \wedge \theta^\nu$. Since $C^A_{AL} = 1$, we have on $\mathcal{G}^*_{hp_\pm} \sim \mathbb{R}_+ \times \mathbb{R}$

$$\omega_\pm = \eta_A\theta^A \wedge \theta^L.$$

The symplectic form ω used in section 2 is obtained from ω_- above through the realization

$$\begin{aligned} A &\mapsto \partial_p & \qquad L &\mapsto p\partial_p - x\partial_x \\ \theta^A &\mapsto dp + pd\ln x & \qquad \theta^L &\mapsto -d\ln x, \end{aligned}$$

and with $\eta_A = -x,\ x \in \mathbb{R}_+$.

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